Exercice 1

1. $f \in C^2(\mathbb{R})$ et $f''(x)=0 \Rightarrow f'(x) = c$

$\Rightarrow f(x) = cx + b \Rightarrow f$ est affine.

2. $f(x) = g(\|x\|)$

$$\frac{\partial f}{\partial x_i}(x) = g'(\|x\|)\,\frac{x_i}{\|x\|}$$

$$\frac{\partial^2 f}{\partial x_i}(x) = g''(\|x\|)\,\frac{x_i^2}{\|x\|^2} + g'(\|x\|)\,\frac{\|x\| - x_i\frac{x_i}{\|x\|}}{\|x\|^2}$$

$$= g''(\|x\|)\,\frac{x_i^2}{\|x\|^2} + g'(\|x\|)\,\frac{1}{\|x\|} - g'(\|x\|)\,\frac{x_i^2}{\|x\|^3}$$

$$\Delta f = \sum_{i=1}^{N} \frac{\partial^2 f}{\partial x_i}(x) = g''(\|x\|) + \frac{N}{\|x\|}\,g'(\|x\|) - g'(\|x\|)\,\frac{1}{\|x\|}$$

$$= g''(\|x\|) + \frac{N-1}{\|x\|}\,g'(\|x\|)$$

donc $g(r)$ vérifie $g''(r) + \frac{N-1}{r}\,g'(r) = 0$ sur $\mathbb{R}_*^+$

$\times\, r^{N-1}$ $\quad r^{N-1}g''(r) + (N-1)\,r^{N-2}g'(r) = 0$

$N > 2$ $\quad \left(r^{N-1}g'(r)\right)' = 0$

$$r^{N-1}g'(r) = C$$

$$g'(r) = \frac{C}{r^{N-1}} \qquad \boxed{g(r) = \frac{C}{r^{N-2}} + C' \qquad N > 2}$$

$N = 2$ $\quad rg''(r) + g'(r) = 0 \Rightarrow (rg') = 0$

$$\Rightarrow rg' = A \Rightarrow \boxed{g(r) = A\ln(r) + C}$$

Conclusion

$$f(\|x\|) = \begin{cases} \dfrac{A}{\|x\|^{N-2}} + C & \text{si } N > 2 \\ A \ln(\|x\|) + C & \text{si } N = 2 \end{cases}$$

Exercice 2.

1. $$\int_{B(x,r+h)} |\nabla u|^2 dx - \int_{B(x,r)} |\nabla u|^2 dx$$

$$= \int_{B(x,r+h) \setminus B(x,r)} |\nabla u|^2 dx$$

$$\frac{1}{h}\Big(E(r+h) - E(r)\Big) = \frac{1}{h} \int_{B(x,r+h) \setminus B(x,r)} |\nabla u|^2 dx$$

$$= \frac{1}{h} \int_r^{r+h} \left(\int_{\partial B(x,s)} |\nabla u|^2 \, d\sigma \right) ds$$

$$= \frac{1}{h} \int_r^{r+h} \int_0^{2\pi} |\nabla u|^2 (s\cos\theta, s\sin\theta) \, s \, ds \, d\theta$$

$$\lim_{h \to 0} \frac{1}{h}\Big[E(r+h) - E(r)\Big] = \int_{\partial B(x,r)} |\nabla u|^2 \, d\sigma$$

$$\int_{B(x_0,r)} \langle \nabla u, \nabla u \rangle dx = \underbrace{\int_{B(x,r)} -\operatorname{div} \nabla u \; u}_{=0} + \int_{\partial B(x,r)} u \nabla u \cdot \nu$$

$$= \int_{\partial B(x,r)} u \frac{\partial u}{\partial \nu} \, d\sigma$$

$$\int_{B(x_0,r)} \Delta u \, dx = \int_{\partial B(x_0,r)} \frac{\partial u}{\partial \nu} \, d\sigma = 0$$

$$E(r) = \int_{\partial B(x_0,r)} u \frac{\partial u}{\partial \nu} \, d\sigma$$

$$= \int_{\partial B(x_0,r)} (u - m_r) \frac{\partial u}{\partial \nu} \, d\sigma$$

$$\leq \frac{1}{2\lambda} \int_{\partial B(x_0,r)} (u - m_r)^2 \, d\sigma + \frac{\lambda}{2} \int_{\partial B(x_0,r)} \left(\frac{\partial u}{\partial \nu}\right)^2 d\sigma$$

$$\leq \frac{r^2}{2\lambda} \int_{\partial B(x_0,r)} \left(\frac{\partial u}{\partial \tau}\right)^2 d\sigma + \frac{\lambda}{2} \int_{\partial B(x_0,r)} \left(\frac{\partial u}{\partial \nu}\right)^2 d\sigma$$

$(\lambda = r)$

$$\leq \frac{r}{2} \int_{\partial B(x_0,r)} \left(\frac{\partial u}{\partial \tau}\right)^2 + \left(\frac{\partial u}{\partial \nu}\right)^2 d\sigma$$

$$\leq \frac{r}{2} E'(r)$$

5. $$\left(\frac{E(r)}{r^2}\right)' = \frac{E'(r) r^2 - E(r) 2r}{r^4}$$

$$= \frac{E'(r) r - 2E(r)}{r^3} \geq 0 .$$

<u>Exercice 3</u>

$$\frac{1}{r^{N-1}} \int_{\partial B(x_0,r)} u(x) \, d\mathcal{H}^{N-1} = \int_{\partial B(x_0,1)} u(rx) \, d\mathcal{H}^{N-1}$$

$$\frac{d}{dr}\left(\int_{\partial B(x_0,1)} u(r x) \, d\mathcal{H}^{N-1} \right) = \int_{\partial B(x_0,1)} \langle \nabla u(rx), x \rangle$$

$$= \frac{1}{r^{N-1}} \int_{\partial B(x_0,r)} \left\langle \nabla u(x), \frac{x}{r} \right\rangle d\mathcal{H}^{N-1}(x)$$

$$= \frac{1}{r^{N-1}} \int_{\partial B(x_0,r)} \frac{\partial u}{\partial \nu}(x)\, d\mathcal{H}^{N-1}(x)$$

2. $$0 = \int_{B(x_0,r)} \Delta u \, dx = \int_{\partial B(x_0,r)} \frac{\partial u}{\partial \nu} \, d\mathcal{H}^{N-1}$$

donc $$\frac{d}{dr}\left(\frac{1}{r^{N-1}} \int_{\partial B(x_0,r)} u(x)\, d\mathcal{H}^{N-1} \right) = 0$$

donc $$\frac{1}{r^{N-1}} \int_{\partial B(x_0,r)} u(x)\, d\mathcal{H}^{N-1} = \text{Constante}$$

or $$\lim_{r\to 0} \left(\frac{1}{r^{N-1} N\omega_N} \int_{\partial B(x_0,r)} u(x)\, d\mathcal{H}^{N-1} \right) = u(x_0) \qquad (*)$$

en effet, $\forall \varepsilon > 0 \ \exists \ \delta > 0, \ |x - x_0| < \delta \Rightarrow |u(x) - u(x_0)| \leq \varepsilon$

$$\left| \frac{1}{N\omega_N \, r^{N-1}} \int_{\partial B(x_0,r)} u(x)\, d\mathcal{H}^{N-1} - u(x_0) \right|$$

$$= \left| \frac{1}{r^{N-1} N\omega_N} \int_{\partial B(x_0,r)} \big(u(x) - u(x_0) \big)\, d\mathcal{H}^{N-1} \right|$$

$$\leq \varepsilon \quad \text{pour} \quad r \leq \delta.$$

donc $(*)$ est OK.

$$\int_{B(x_0,r)} u(x)\,dx = \int_0^r \int_{\partial B_t} u(x)\, d\mathcal{H}^{N-1}(x)\, dt$$

$$\frac{1}{\omega_N r^N} \int_{B(x_0,r)} u(x) = \int_0^r \frac{1}{\omega_N r^N} \int_{\partial B_t} u(x)\, d\mathcal{H}^{N-1}(x)\, dt$$

$$= \int_0^r \frac{N}{r^N} \frac{t^{N-1}}{N\omega_N t^{N-1}} \int_{\partial B_t} u(x)\, d\mathcal{H}^{N-1}(x)\, dt$$

$$= \frac{N}{r^N} \int_0^r t^{N-1}\, u(x_0)\, dt$$

$$= u(x_0) \frac{N}{r^N} \left[\frac{t^N}{N} \right]_0^r = u(x_0)$$

3. si $\Delta u = 0$ alors $\Delta \partial_{x_i} u = 0$

donc $$\frac{\partial}{\partial x_i} u(x) = \frac{1}{\cancel{N\omega_N R^{N-1} \int_{\partial B(x,R)} \frac{\partial u}{\partial x_i} d\mathcal{H}^{N-1}}}$$

$$= \frac{1}{\omega_N R^N} \int_{B(x,R)} \frac{\partial}{\partial x_i} u(x)\, dx$$

$$= \frac{1}{\omega_N R^N} \int_{\partial B(x,R)} u\, \nu_i\, d\mathcal{H}^{N-1}(x)$$

$$\left| \frac{\partial}{\partial x_i} u(x) \right| \leq \frac{1}{\omega_N R^N} \int_{\partial B(x,R)} |u|\, d\mathcal{H}^{N-1}(x)$$

$$\leq \frac{1}{\omega_N R^N} \|u\|_\infty \underbrace{\mathcal{H}^{N-1}(\partial B(x_0,R))}_{=N\omega_N R^{N-1}}$$

$$\leq \frac{1}{\omega_N R} \|u\|_\infty$$

4. suffit de faire $R \to 0$ -

Exercice 4 :

1. $v|_{H^+}$ et $v|_{H^-}$ sont C^1.

$$\lim_{y\to 0^+} \frac{\partial v}{\partial x}(x,y) = \lim_{y\to 0^+} \frac{\partial}{\partial x} u(x,y) = \frac{\partial u}{\partial x}(x,0)$$
$$= 0$$

car $u|_{\{x=0\}}$ est constante

$$\lim_{y\to 0^-} \frac{\partial v}{\partial x}(x,y) = -\frac{\partial u}{\partial x}(x,0) = 0$$ donc OK.

$$\lim_{y\to 0^+} \frac{\partial v}{\partial y}(x,y) = \lim_{y\to 0^+} \frac{\partial u}{\partial y}(x,y) = \frac{\partial u}{\partial y}(x,0)$$

$$\lim_{y\to 0^-} \frac{\partial v}{\partial y}(x,y) = + \frac{\partial}{\partial y}\left(u(x,-y)\right) = \frac{\partial u}{\partial y}(x,0)$$

donc OK.

2. On prend une fonction test $\varphi(x,y)$

$$\left\langle \frac{\partial^2}{\partial x} v, \varphi \right\rangle = \int_{\mathbb{R}^2} v(x,y) \frac{\partial^2}{\partial x} \varphi(x,y) \, dx\, dy$$

$$= \int_{H^+} u(x,y) \frac{\partial^2}{\partial x} \varphi(x,y) + \int_{H^-} -u(x,-y) \frac{\partial^2}{\partial x} \varphi(x,y)$$

$$= \int_{H^+} u(x,y) \frac{\partial^2}{\partial x} \varphi(x,y) + \int_{H^+} -u(x,y) \frac{\partial^2}{\partial x} \varphi(x,-y)$$

$$= \int_{H^+} \frac{\partial^2}{\partial x} u(x,y) \, \varphi(x,y) + \int_{H^+} -\frac{\partial^2}{\partial x} u(x,y) \, \varphi(x,-y)$$

$$\left\langle \frac{\partial^2}{\partial y} v, \varphi \right\rangle = \int_{\mathbb{R}^2} v(x,y) \frac{\partial^2}{\partial y} \varphi(x,y)$$

$$= \int_{H^+} u(x,y) \frac{\partial^2}{\partial y} \varphi(x,y) + \int_{H^-} -u(x,-y) \frac{\partial^2}{\partial y} \varphi(x,y)$$

$$= \int_{-R}^{R} \int_{0}^{R} u(x,y) \frac{\partial^2}{\partial y} \varphi(x,y) + \int_{-R}^{R} \int_{-R}^{0} -u(x,-y) \frac{\partial^2}{\partial y} \varphi(x,y)$$

$$= -\int_{-R}^{R} \int_{0}^{R} \frac{\partial}{\partial y} u \, \frac{\partial}{\partial y} \varphi + \int_{-R}^{R} \underbrace{- u(x,0) \frac{\partial}{\partial y} \varphi(x,0)}_{=0}$$

$$+ -\int_{-R}^{R} \int_{-R}^{0} \partial_y u(x,-y) \frac{\partial}{\partial y} \varphi(x,y)$$

$$= \int_{-R}^{R} \int_{0}^{R} \frac{\partial^2}{\partial y} u(x,y) \, \varphi(x,y) + \int_{-R}^{R} - \partial_y^2 u(x,-y) \, \varphi(x,y)$$

$$+ \cancel{\int_{-R}^{R} \frac{\partial}{\partial y} u(x,0)} \, \varphi(x,0)$$

$$- \cancel{\int_{-R}^{R} \partial_y u(x,0)} \, \varphi(x,0)$$

$$= \int_{H^+} \partial_y^2 u \left[\varphi(x,y) - \varphi(x,-y) \right]$$

$\Rightarrow$ OK