Feuille 2 (Distrib)

Exercice 1

1. oui. $\langle T, \varphi + \psi \rangle = \langle T, \varphi \rangle + \langle T, \psi \rangle$

$$|\langle T, \varphi \rangle| \leq \|\varphi^{(n)}\|_\infty \int_0^1 x^n \, dx$$
$$\leq C \, p_n(\varphi).$$

2. Non (pas linéaire).

Exercice 2

1. $$\langle H, \varphi \rangle = \int_{\mathbb{R}} \mathbb{1}_{[0,+\infty[} \varphi(x) \, dx$$
$$= \int_0^{+\infty} \varphi(x) \, dx$$

$$|\langle H, \varphi \rangle| \leq |\operatorname{supp} \varphi| \, \|\varphi\|_\infty$$
$$\leq C_K \, p_0(\varphi)$$

(bien linéaire)

$$\langle H', \varphi \rangle = -\langle H, \varphi' \rangle = -\int_0^{+\infty} \varphi'(x) \, dx$$
$$= \varphi(0) = \langle \delta_0, \varphi \rangle$$

$$\langle H^{(n)}, \varphi \rangle = (-1)^n \langle H, \varphi^{(n)} \rangle = (-1)^n \varphi^{(n-1)}(0)$$

$$H^{(n)} = (-1)^n \delta_0^{(n-1)}$$

2. $\langle |x|', \varphi \rangle = -\langle |x|, \varphi' \rangle$

$$= -\int_{\mathbb{R}} |x|\, \varphi'\, dx$$

$$= \int_{-\infty}^{0} x\, \varphi'(x) + -\int_{0}^{+\infty} x\, \varphi'\, dx$$

$$= -\int_{-\infty}^{0} \varphi + \cancel{[x\varphi]_{-\infty}^{0}} + \int_{0}^{+\infty} \varphi\, dx$$

$$= \langle 1\!\!1_{[0,+\infty[} - 1\!\!1_{]-\infty,0[}, \varphi \rangle$$

3. $|\langle T, \varphi \rangle| \leq C\, p_K(\varphi)$

$$|\langle T', \varphi \rangle| = |\langle T, \varphi' \rangle| \leq C\, p_K(\varphi') \leq C\, p_{K+1}(\varphi)$$

(prendre $f \in C^1$)

4. supposons $\exists f \in C^1$ / $\langle \delta_0, \varphi \rangle = \int_{\mathbb{R}} f \varphi$

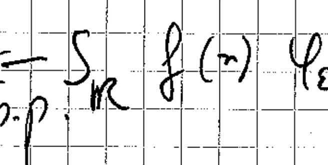
$\forall \varphi \in \mathcal{D}(\mathbb{R})$, $\varphi(0) = \int_{\mathbb{R}} f\, \varphi\, dx$

Soit $\varphi_\varepsilon \in C^\infty$ fonction approximation de l'unité

$$f(y) \xleftarrow[p.p]{} \int_{\mathbb{R}} f(x)\, \varphi_\varepsilon(y-x)\, dx = \varphi_\varepsilon(y) \longrightarrow \begin{cases} +\infty & \text{si } y = 0 \\ 0 & \text{si } y \neq 0 \end{cases}$$

absurde.

5. ~~δ_0, ...~~ $|\varphi'(0)| \leq C\, \|\varphi\|_\infty$



Exercice 3 :

$$f(x) = f(0) + \int_0^x f'(t)\,dt$$

$$= \cancel{\int_0^x f'(t)\,dt}$$

$$= \cancel{\left[t f''(t)\right]_0^x - \int_0^x t f''(t)\,dt}$$

$$f(x) = \int_0^x f'(t)\,dt \underset{xu = t}{=} x \underbrace{\int_0^1 f'(xu)\,du}_{\in C^\infty \text{ (intégrale à paramètre)}}$$

si $f'(0) = 0$

$$f(x) = \int_0^x f'(t)\,dt$$

$$= \left[(t-x) f'(t)\right]_0^x - \int_0^x (t-x) f''(t)\,dt$$

$$= -\int_0^x (t-x) f''(t)\,dt$$

$$= -x^2 \int_0^1 (u-1) f''(ux)\,du$$

$$= x^2 \psi(x).$$

Exercice 4 :

1. $$\int_{|x|>\varepsilon} \frac{\varphi(x)}{x}\,dx = \int_{-M}^{-\varepsilon} \frac{\varphi(0)}{x}\,dx + \int_{\varepsilon}^{M} \frac{\varphi(0)}{x}\,dx + \underbrace{\int_{|x|>\varepsilon} \frac{\varphi-\varphi(0)}{x}}_{\substack{\text{OK} \\ \text{(Lemme Hadamard)}}}$$

$$= \varphi(0)\left[\cancel{\ln(M)} - \ln(\varepsilon)\right] + \varphi(0)\left[\cancel{\ln(\varepsilon)} - \ln(M)\right] \Rightarrow \text{OK}.$$

2. linéaire OK.

$$\varphi - \varphi(0) = x\int_0^1 \varphi'(xu)\,du$$

$$|\varphi(x) - \varphi(0)| \leq |x| \quad P_1(\varphi)$$

$$\left|\langle vp(\tfrac{1}{x}), \varphi\rangle\right| = \left|\int_{|x|>\varepsilon} \frac{\varphi-\varphi(0)}{x}\right| \leq C(\mathrm{supp}\,\varphi)\; P_1(\varphi).$$

3. $$\langle x\,vp(\tfrac{1}{x}), \varphi\rangle = \langle vp(\tfrac{1}{x}), x\varphi\rangle$$

$$= \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} \varphi(x)\,dx = \langle 1, \varphi\rangle$$

4. $\ln|x| \in L^1_{loc}$

$$\langle \ln|x|', \varphi\rangle = -\int_{\mathbb{R}} \ln|x|\,\varphi'\,dx$$

$$= \lim_{\varepsilon\to 0} -\int_{\mathbb{R}\setminus]-\varepsilon,\varepsilon[} \ln|x|\,\varphi'(x)\,dx$$

$$- \int_{-\infty}^{-\varepsilon} \ln|x| \varphi'(x) - \int_{\varepsilon}^{+\infty} \ln|x| \varphi'(x)\,dx$$

$$- \left[\ln|x| \varphi \right]_{-\infty}^{-\varepsilon} + \int_{-\infty}^{-\varepsilon} + \frac{1}{x} \varphi(x)\,dx + - \left[\ln|x| \varphi \right]_{\varepsilon}^{+\infty} + \int_{\varepsilon}^{+\infty} \frac{\varphi(x)}{x}$$

$$- (\ln \varepsilon) \varphi(-\varepsilon) + \ln \varepsilon \, \varphi(\varepsilon) \qquad \langle vp\left(\tfrac{1}{x}\right), \varphi \rangle$$

$$\ln(\varepsilon) \Big(\varphi(\varepsilon) - \varphi(-\varepsilon) \Big) \longrightarrow 0.$$

$$= \cancel{\varphi(0)} + \varepsilon \varphi'(0) + o(\varepsilon)$$
$$- \cancel{\varphi(0)} + \varepsilon \varphi'(0) + o(\varepsilon)$$

5. a) Lemme Hadamard à $\varphi(x) - \varphi(0)\, \theta(x)$

b) $xT = 0 \implies \langle T, \varphi - \varphi(0)\theta \rangle = 0$

$$\implies \langle T, \varphi \rangle = \langle T, \varphi(0)\, \theta \rangle$$
$$= \varphi(0) \underbrace{\langle T, \theta \rangle}_{= c}$$
$$= \langle c\delta_0, \varphi \rangle$$

c) on sait que $x\, vp\left(\frac{1}{x}\right) = 1$

$$xT = 1 \iff T = c\delta_0 + vp\left(\frac{1}{x}\right).$$

6. Pas unicité car $xT = S \implies x(T + \delta_0) = S$

on définit $\langle T, \varphi \rangle = \langle S, \psi \rangle$

où ψ est la fonction telle que $\varphi = \varphi(0)\theta + x\psi$

$$\psi := \left(\varphi - \varphi(0)\theta\right)\frac{1}{x}$$

$$\langle T, \varphi \rangle := \left\langle S, \frac{1}{x}\left(\varphi - \varphi(0)\theta\right) \right\rangle$$

bien une distribution (linéaire en φ) et continue.

de plus $\langle xT, \varphi \rangle = \langle T, x\varphi \rangle = \langle S, \varphi \rangle$

car $\frac{1}{x}\left(x\varphi - (x\varphi)|_{(x=0)}\,\theta\right) = \varphi$.

7. $xT = 1 \iff T = c\,\delta_0 + vp\left(\frac{1}{x}\right)$

$xT = \delta_0 \iff T = c\,\delta_0 + \delta_0'$

$xT = vp\left(\frac{1}{x}\right) \iff T = c\,\delta_0 + pf\left(\frac{1}{x^2}\right)$? (ex 5.)

Exercice 6.

(a) $\langle \sin(nx), \varphi \rangle \xrightarrow[n \to +\infty]{} 0$ Riemann-Lebesgue

(b) $\left\langle \frac{\sin(nx)}{x}, \varphi \right\rangle = \int_{\mathbb{R}} \frac{\sin(nx)}{x}\varphi(x)\,dx$

$y = nx$, $dy = n\,dx$ $= \int_{\mathbb{R}} \frac{\sin(y)}{y}\, n\,\varphi\left(\frac{y}{n}\right)\frac{dy}{n}$

$\left|\frac{\sin(y)}{y}\right| \in L^1_{loc}$ pour M fixé.

$$\int_{-M}^{M} \frac{\sin(y)}{y}\varphi\left(\frac{y}{n}\right)dy \longrightarrow \varphi(0)\int_{-M}^{M}\frac{\sin(y)}{y}$$

C.V dominée

$$\int_{\Pi}^{+\infty} \frac{\sin(y)}{y}\, \varphi\left(\frac{y}{n}\right) dy$$

$\mathrm{Supp}(\varphi) \subset [-L, L]$

$\mathrm{Supp}\left(\varphi\left(\frac{y}{n}\right)\right) \subset [-nL, nL]$

$$= \int_{\Pi}^{nL} \frac{\sin(y)}{y}\, \varphi\left(\frac{y}{n}\right) dy$$

$$\cancel{= -\int_{\Pi}^{nL} \cos(y) \frac{1}{y^2} \varphi\left(\frac{y}{n}\right) dy + \left[\cos(y)\varphi\left(\frac{y}{n}\right)\frac{1}{y^2}\right.}$$

$$= -\int_{\Pi}^{nL} (-\cos(y)) \left(\frac{1}{y}\varphi\left(\frac{y}{n}\right)\right)' dy$$

$$+ \left[-\cos(y)\, \frac{1}{y}\varphi\left(\frac{y}{n}\right)\right]_{\Pi}^{nL}$$

$$= -\int_{\Pi}^{nL} (-\cos(y)) \left(\frac{1}{y}\varphi\left(\frac{y}{n}\right)\right)' dy + \cos(\Pi)\frac{1}{\Pi}\varphi\left(\frac{\Pi}{n}\right)$$

$$= -\int_{\Pi}^{nL} (-\cos(y)) \left(\frac{-1}{y^2}\right)\varphi\left(\frac{y}{n}\right) dy + \cos(\Pi)\frac{1}{\Pi}\varphi\left(\frac{\Pi}{n}\right)$$

$$- \int_{\Pi}^{nL} -\cos(y)\frac{1}{y}\frac{1}{n}\underbrace{\varphi'\left(\frac{y}{n}\right)}_{\leq C} dy$$

$\frac{\cos(y)}{y}$ <u>intégrable</u>

$\longrightarrow 0$

~~$\Pi \to +\infty$ puis~~ $n \to +\infty$ puis $\Pi \to +\infty$.

<u>OK</u>

(c) $\int n\sin(nx)\,\mathbb{1}_{\mathbb{R}_+}\varphi(x) = \int_0^{+\infty} n\sin(nx)\,\varphi(x)\,dx$

$$= \int_0^{+\infty} \cancel{n}\sin(y)\,\varphi\left(\frac{y}{n}\right)\frac{dy}{\cancel{n}}$$

$y = nx$

$$= \int_0^{M} \sin(y)\,\varphi\left(\frac{y}{n}\right)dy \longrightarrow \varphi(0)\int_0^{M}\sin(y)$$

$$\int_0^{+\infty} n\sin(nx)\,\varphi(x)\,dx = \left[n\,\frac{-\cos(nx)}{n}\,\varphi(x)\right]_0^{\infty} = \varphi(0) - \underbrace{\int_0^{+\infty} -\cos(nx)\,\varphi'(x)\,dx}_{\xrightarrow[n\to\infty]{} 0}$$

$$n\sin(nx) \xrightarrow{\mathcal{D}'} \delta_0$$

(d) $n\left(\varphi\left(\frac{1}{n}\right) - \varphi\left(-\frac{1}{n}\right)\right) = n\left(\cancel{\varphi(0)} + \varphi'(0)\frac{1}{n} - \cancel{\varphi(0)} + \frac{1}{n}\varphi'(0)\right)$

$$\longrightarrow \boxed{2\varphi'(0)}$$

$\downarrow \mathcal{D}'$

$2\delta_0' -$