Ex 2: $f(x) = \int_a^x g(t)\,dt$ $\qquad g \in L^1(]a,b[)$

- bien défini

soit $\varphi \in \mathcal{D}(]a,b[)$

$-\langle f', \varphi \rangle$

$$\int_a^b f(x)\,\varphi'(x)\,dx = \int_a^b \left(\int_a^x g(t)\,\varphi'(x)\,dt \right) dx$$

$$= \int_a^b \int_a^b \left(\mathbb{1}_{]a,x[}(t)\, g(t)\, \varphi'(x)\, dt \right) dx$$

$$= \int_a^b g(t) \left(\int_a^b \mathbb{1}_{]a,x[}(t)\, \varphi'(x)\, dx \right) dt$$

$$= \int_a^b g(t) \left(\int_t^b \varphi'(x)\,dx \right) dt = -\int_a^b g(t)\,\varphi(t)$$

$$= -\langle g, \varphi \rangle$$

on a démontré $\langle f', \varphi \rangle = \langle g, \varphi \rangle$

$\forall \varphi$, donc $f' = g$ dans $\mathcal{D}'(]a,b[)$

$g \in L^p$ et $f \in L^p$ $(f \in L^\infty)$

donc $f \in W^{1,p}(]a,b[)$

2. $f(x) = \int_a^x g(t)\,dt$ $\quad p > 1$

$$|f(y) - f(x)| = \left|\int_x^y g(t)\,dt\right|$$

$$\leq \int_x^y 1 \times |g(t)|\,dt \leq |y-x|^{1/p'} \times \left(\int |g|^p\right)^{1/p}$$

exercice 3.

$$f(x) = \left[- \ln(\|x\|^2) \right]^{\alpha}$$

$x \in B_{1/2}(0)$

$$= |\ln(\|x\|^2)|^{\alpha} = 2^{\alpha} |\ln(\|x\|)|^{\alpha}$$

question : $f \in W^{1,2}(B_{1/2}(0))$

$f \in L^2$?

$$\int_{B_{1/2}} |\ln(\|x\|)|^{2\alpha} \, dx$$

$$= \int_0^{1/2} \left(\int_{\partial B_t} |\ln(t)|^{2\alpha} \, d\sigma \right) dt$$

$$= \int_0^{1/2} |\ln(t)|^{2\alpha} \, t^{n-1} \, dt$$

$n > 1$
$\Rightarrow$ OK

intégrale de Bertrand $\displaystyle\int_0^{1/2} \frac{1}{|\ln(t)|^a\, t^b}$

intégrable si $b < 1$

ou $b = 1$ et $a > 1$

$f \in L^2$

$$f(x) = \left(-\ln(\|x\|^2)\right)^\alpha$$

g ? $\partial_{x_i} f = g_i$ dans $\mathcal{D}'(B_{1/2})$

pour $x \neq \{0\}$ on a :

$$\partial_{x_i} f(x) = -\alpha\left(-\ln(\|x\|^2)\right)^{\alpha-1} \frac{2x_i}{\|x\|^2} = g_i(x)$$

$$\|g_i\|_2^2 = \int_{B_{1/2}} c\,|\ln(\|x\|)|^{2(\alpha-1)} \frac{|x_i|^2}{\|x\|^4} \leq c \int_{B_{1/2}} |\ln(\|x\|)|^{2(\alpha-1)} \frac{1}{\|x\|^2}$$

$$\leq c \int_0^{1/2} \ln(t)^{2(\alpha-1)} \times \frac{1}{t^2}\, t^{N-1}\, dt$$

$$\leq c \int_0^{1/2} \frac{1}{\ln(t)^{(1-\alpha)\times 2}\, t^{2+1-N}}\, dt$$

si $2+1-N < 1 \Rightarrow \boxed{2 < N \Rightarrow \text{OK}}$

ou $\underbrace{2+1-N=1}_{N=2}$ et $2(1-\alpha) > 1$

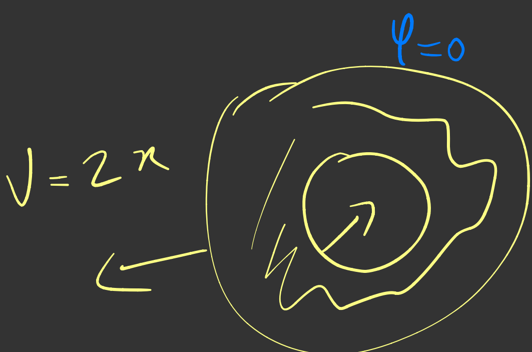
$\boxed{\frac{1}{2} > \alpha}$

question : $\partial_{x_i} f = g_i$ dans $\mathcal{D}'$?

$$\int_{B_{1/2}} f(x)\,\partial_{x_i}\varphi\,dx = \int_{B_{1/2}\setminus B_\varepsilon} + \underbrace{\int_{B_\varepsilon} f(x)\,\partial_{x_i}\varphi}_{\longrightarrow 0}$$

$\varepsilon \to 0$ car $f \in L^1_{loc}$

$$\int_{B_{1/2}\setminus B_\varepsilon} f(x)\,\partial_{x_i}\varphi\,dx = \underbrace{-\int_{B_{1/2}\setminus B_\varepsilon} \partial_{x_i} f(x)\,\varphi(x)\,dx}_{\longrightarrow -\int_{B_{1/2}} g_i\varphi} + \underbrace{\cancel{\int_{\partial B_{1/2}} f\varphi\,\nu_i\,d\sigma}}_{=0} - \underbrace{\int_{\partial B_\varepsilon} f\varphi\,\frac{x_i}{\varepsilon}\,d\sigma}_{?}$$



$$\int_{\partial B_\varepsilon} \left| f \varphi \frac{x_i}{\varepsilon} \right| d\sigma \leq \|\varphi\|_{L^\infty} \int_{\partial B_\varepsilon} |\ln(\varepsilon)|^\alpha \times 1 \, d\sigma$$

$$\leq \|\varphi\|_\infty |\ln(\varepsilon)|^\alpha \varepsilon^{N-1}$$

$$\boxed{N \geqslant 2} \qquad \xrightarrow[\varepsilon \to 0]{} 0$$

<u>Conclusion</u> : pour $N \geqslant 2$, $\partial_{x_i} f = g_i$ dans $\mathcal{D}'(B_{1/2})$