Lecture notes on
Regularity theory for a class of free-discontinuity problems arising in physical mechanics, from Mumford-Shah to Griffith

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Abstract

These notes are coming from a summer course that I gave in Beihang university, Beijin, China on 15th-22th june 2019.

The aim of this course is to expose some recent developments about the regularity of minimizers of the Griffith functional, especially the C^1 regularity result in 2D contained in [JFFA19]. Since this functional is related to the classical Mumford-Shah functional, we will therefore review some classical facts about the Mumford-Shah functional as well.

Warning: the following notes were written fairly quickly with no careful checking. Moreover, they are not exactly identical to what was really given during the lectures, sometimes with more elements but sometimes with less. I hope however that the material contained in these notes, especially the bibliography part, could help anyone who attended the course to go further on this subject. If the reader needs more details about some parts, or for any other question regarding to these notes, please feel free to write me an e-mail at: *lemenant@univ-paris-diderot.fr*

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1 Lecture day ♯1: Presentation of the problems and existence issues.

1.1 The Classical Mumford-Shah Problem

The Classical Mumford-Shah functional was originally cast in dimension 2 in order to solve an image segmentation problem. In the simplest setting a given image is a L^∞ function $g : \Omega \to \mathbb{R}$ defined on a bounded planar domain $\Omega \subset \mathbb{R}^2$, and one wants to find a 1-dimensional set representing its significative jump points, where we expect the edges of the image to lie. To do so, in 1989 Mumford and Shah [MS89] proposed to minimize the following functional

$$J(u,K) = \int_\Omega |u-g|^2 dx + \int_{\Omega\setminus K} |\nabla u|^2 dx + \mathcal{H}^1(K) \tag{1}$$

among all pairs $(u,K) \in \mathcal{A}(\Omega)$ where

$$\mathcal{A}(\Omega) = \{(u,K)\ ;\ K \subset \Omega \text{ is closed and } u \in W^{1,2}(\Omega \setminus K)\}.$$

Actually, minimizing the functional produces two objects : the set K which represents the edges of the image (usually called the "singular set"), and at the same time a function u which is smooth outside this set K, and that is very close to the original image in the L^2 norm. The first term of the functional is here to guarantee the latter fact. This term can be considered as a "Dirichlet condition", say, and does not count much in view of the regularity theory. The main terms of the functional are the second and third terms, which work together as two competitive terms: if for instance, the image g has a sharp significant jump somewhere, in other words if there is an edge in the image, then in the minimizing process a piece of set K would be quite useful to be added in order to save some gradient of u. But the price to pay is comparable to the length of the added set.

The functional works pretty well in practice. A numerical method can be obtained using the phase-field approximation of Ambrosio-Tortorelli [AT92, Bou99, BC94].

In the same paper [MS89], Mumford and Shah conjecture the following.

Conjecture 1 (Mumford and Shah conjecture 1989)**.** *Let (u,K) be a reduced minimizer of the functional J. Then K must be a finite union of $C^{1,\alpha}$ arcs.*

The conjecture is still open even though Bonnet has almost proved it in 1996 [Bon96]. We will try to give some more detail in Section 3. The $C^{1,\alpha}$ regularity here is not sharp, but it is just a first step to prove further regularity by standard elliptic theory. Up to now, what is still really missing is the finite number of curves.

Let us spend some time now about the proof of existence for a minimizer.

1.2 Short review on SBV theory and weak existence for Mumford-Shah

The existence of a minimizer can be obtained by relaxing the functional in a more general space called SBV. Let us give here a definition.

For any open set $\Omega \subset \mathbb{R}^N$, the space $BV(\Omega)$ is the class of all functions $u \in L^1_{loc}(\Omega, \mathbb{R})$ such that Du (the derivative of u in the distributional sense) is a finite measure. If $u \in (BV(\Omega))^N$ is a BV vector field, a point $z \in \mathbb{R}^N$ is an approximate limit for u at point x if

$$\lim_{\rho \to 0} \frac{1}{|B(x,\rho)|} \int_{B(x,\rho)} |u(y) - z| dy = 0.$$

The set S_u of points where this property does not hold is called the *approximate discontinuity set* of u, and the points z for which the limit exists is called an approximative limit of u at point x and is denoted by $\tilde{u}(x)$. A remarkable result of Federer and Vol'pert (see [AFP00, Th. 3.78.]) says that when $u \in (BV(\Omega))^N$, then S_u is $(N-1)$-rectifiable and $D^s u$ (the singular part of Du with respect to $\mathcal{L}^N$ in the Radon-Nikodym decomposition $Du = D^a u + D^s u$ restricted to S_u is absolutely continuous with respect to $\mathcal{H}^{N-1}$. We will say that $u \in (SBV(\Omega))^N$ when $D^s u$ is actually *concentrated* on S_u, which means that $D^s u(\mathbb{R}^N \setminus S_u) = 0$.

The density of the regular part $D^a u$ of Du with respect to $\mathcal{L}^N$, denoted by ∇u, coincides $\mathcal{L}^N$-a.e. with the *approximate differential* of u (see [AFP00, Th. 3.83.]). A function u is approximately differentiable at the Lebesgue point x if there exists a matrix $\nabla u(x)$ such that

$$\lim_{\rho \to 0} \frac{1}{|B(x,\rho)|} \int_{B(x,\rho)} \frac{|u(y) - \tilde{u}(x) - \nabla u(x).(y-x)|}{\rho} dy = 0.$$

In the sequel we will also use the notion of *trace* of u on the singular set S_u. Since S_u is rectifiable, one can fix an orientation $\nu_u : S_u \to \mathbb{S}^{N-1}$ in such a way that for $\mathcal{H}^{N-1}$-a.e. $x \in S_u$ the approximate tangent plane to S_u at x is orthogonal to the vector $\nu_u(x)$. Then for any $x \in S_u$ and $\rho > 0$ we define $B(x,\rho)^+ := B(x,\rho) \cap \{y; \langle y, \nu_u(x)\rangle \geq 0\}$ and $B(x,\rho)^- := B(x,\rho) \cap \{y; \langle y, \nu_u(x)\rangle \leq 0\}$. For $\mathcal{H}^{N-1}$-a.e. $x \in S_u$, Theorem 3.77. of [AFP00] provides the existence of traces $u^+(x)$ and $u^-(x)$ satisfying

$$\lim_{\rho \to 0} \frac{1}{|B(x,\rho)^\pm|} \int_{B(x,\rho)^\pm} |u(y) - u^\pm(x)| dy = 0. \tag{2}$$

The set of points $x \in S_u$ where $u^\pm(x)$ exist is called the *jump set* and is denoted by J_u. It can be shown that $\mathcal{H}^{N-1}(S_u \setminus J_u) = 0$ and for $x \in J_u$ the quantity $(u^+(x) - u^-(x))$ is called the jump of u at point x, whose sign depends on the orientation of S_u. Moreover for any $u \in (SBV(\Omega))^N$ the representation

$$Du = D^a u + D^s u = \nabla u \, \mathcal{L}^N + (u^+ - u^-) \otimes \nu_u \, \mathcal{H}^{N-1} | S_u, \tag{3}$$

holds.

Let us also mention that a simple approximation argument [AFP00, Proposition 4.4.] says that, if $K \subset \Omega$ is closed and $\mathcal{H}^{N-1}(K) < +\infty$, then any $u \in L^\infty(\Omega) \cap W^{1,1}(\Omega \setminus K)$ belongs to $SBV(\Omega)$ and $\mathcal{H}^{N-1}(S_u \setminus K) = 0$.

The existence of minimizers was first proved by De Giorgi, Carriero, and Leaci in [DGCL89]. Their strategy is to relax the functional on SBV, i.e. to consider

$$\tilde{J}(u) = \int_\Omega |u-g|^2 dx + \int_\Omega |\nabla u|^2 dx + \mathcal{H}^1(S_u), \quad u \in SBV(\overline{\Omega}),$$

where S_u is the singular set of u and ∇u is the approximate gradient of u (we refer to [AFP00] for the definition of SBV). It is quite easy to show that $\tilde{J}$ admits some minimizers due to the compactness result of Ambrosio [Amb89]. It is also not difficult to see that if $(u, K) \in \mathcal{A}$ is such that $\mathcal{H}^1(K) < +\infty$, then $u \in SBV(\Omega)$ (using [AFP00, Proposition 4.4]), and $J(u, K) \geq \tilde{J}(u)$ thus

$$\inf_{(u,K)\in\mathcal{A}} J(u, K) \geq \min_{u\in SBV(\Omega)} \tilde{J}(u). \tag{4}$$

1.3 Strong existence: Theorem of De Giorgi Carriero Leaci

The issue in [DGCL89] is then to prove the reverse inequality in (4), and this is obtained by showing that for any minimizer $u \in SBV(\Omega)$ of $\tilde{J}$, the singular set S_u is essentially closed, namely that $\mathcal{H}^1(\overline{S_u} \setminus S_u) = 0$.

Indeed, then $(u, \overline{S_u}) \in \mathcal{A}$ and therefore

$$\inf_{(w,K)\in\mathcal{A}} J(w, K) \leq J(u, \overline{S_u}) = \tilde{J}(u) = \min_{v\in SBV} \tilde{J}(v).$$

The first ingredient in the proof of [DGCL89] is the following lemma.

Lemma 1. *If $u \in SBV(\Omega)$, then S_u is always contained in the complement of the set Ω_0 of points $x \in \Omega$ for which*

$$\lim_{r\to 0} \frac{1}{r}\Big(\int_{B_r(x)} |\nabla u|^2\, dx + \mathcal{H}^1(S_u \cap B_r(x))\Big) = 0. \tag{5}$$

This follows from a now famous Poincaré type estimate on SBV functions (one can control a truncation of u minus a median by the integral of the gradient of u, provided the jump set is small enough).

The second ingredient is a compactness argument which provides the existence of $\varepsilon_0 > 0$ for which

$$\frac{1}{r}\Big(\int_{B_r(x)} |\nabla u|^2\, dx + \mathcal{H}^1(S_u \cap B_r(x))\Big) \leq \varepsilon_0 \quad \Longrightarrow \quad (5).$$

While the first fact holds for any SBV function, the second one needs u to be a minimizer. From the above two facts it can be proved that Ω_0 must be open, and that $\overline{S_u} = \Omega \setminus \Omega_0$. Then a standard argument from measure theory says that (5) must hold $\mathcal{H}^1$-a.e. on $\Omega \setminus S_u$ [DGCL89, Lemma 2.6] thus in conclusion we have that $\mathcal{H}^1(\overline{S_u} \setminus S_u) = 0$.

Let us mention that an alternative proof of existence of minimizers without using the theory of SBV functions is proposed in [DMMS92] (see also [Dav05, Section 36]).

Finally, two other different and very recent proofs are proposed in [BL14] in any dimension and in [LF13] in dimension 2 (see also Section **??**).

1.4 Griffith functional

1.4.1 The propagation of Fracture

According to Griffith's theory, the propagation of a brittle fracture in an elastic body is governed by the competition between the energy spent to produce a crack, proportional to its length, and the corresponding release of bulk energy. An energetic formulation of this idea is the core of variational models for crack propagation, which were introduced by Francfort and Marigo in [GAF98] and are based on a Mumford-Shah-type functional.

If $\Omega \subset \mathbb{R}^N$ (usually $N = 3$, sometimes $N = 2$ for simplicity) is the reference configuration of an elastic body subject to a displacement $u : \Omega \to \mathbb{R}^N$ with prescribed boundary datum $u = g$ on $\partial\Omega$, the elastic energy is given by

$$\frac{1}{2}\int_{\Omega} \mathbf{A}e(u) : e(u)\, dx, \tag{6}$$

where $e(u) = \frac{1}{2}(Du + Du^T)$ is the symmetrical part of the gradient of u, the notation " : " denotes the usual scalar product on matrices, and $\mathbf{A}$ is the fourth order Hooke's tensor

$$\mathbf{A}e = \lambda Tr(e) Id + 2\mu e.$$

The constants $\lambda > 0$ and $\mu > 0$ are the so-called Lamé coefficients, and minimizers of the "Dirichlet type" energy (6) are solutions to an elliptic system called the Lamé system. For a given crack $K \subset \Omega$, the value of

$$E(K, g) := \min_{u \in LD(\Omega \setminus K); u = g \text{ on } \partial\Omega} \frac{1}{2}\int_{\Omega} \mathbf{A}e(u) : e(u)\, dx, \tag{7}$$

is called the bulk energy (the space LD being those of $u \in L^2$ with $e(u)$ in L^2).

There is a particular case, called "anti-plane shear", where the energy (6) reduces to the classical Dirichlet energy. This happens when the domain is a cylinder $\Omega \times \mathbb{R}$, with $\Omega \subset \mathbb{R}^2$, and assuming the crack to be vertically invariant, while the displacement is vertical only. Under those assumptions, the problem reduces to a purely 2D scalar problem, and the energy involved reduces to the classical Mumford-Shah energy. It is often useful to reduce to this simpler case for which the tools from the Mumford-Shah functional can directly apply.

But one of the main difference with the original Mumford-Shah problem, is that the growth of a crack in an elastic body is an evolution process.

For simplicity we now restrict ourselves exclusively to the case $N = 2$. The idea of Francfort and Marigo is to consider, for a given time-dependent loading process $g(t)$ on $\partial\Omega$, the quasi-static evolution of the Mumford-Shah type energy

$$\mathcal{G}(u, K) := \frac{1}{2}\int_{\Omega \setminus K} \mathbf{A}e(u) : e(u)\, dx + \kappa \mathcal{H}^1(K), \tag{8}$$

where the constant $\kappa > 0$ is related to the toughness of the material.

The functional (10) looks like a simple variant of the standard Mumford-Shah functional, but it is just a foggy analogy since most of the desired regularity results are still unknown. Not even the starting point of the regularity theory, that is, the density lower boun. In other words there is no analogue for this functional, to the famous De Giorgi-Carriero-Leaci paper [DGCL89]. Any C^1 regularity result would be also welcome but this looks out of reach for the moment.

Now, the construction of the evolution proceeds as follows: first discretize the time line via $0 < t_1 < \cdots < t_k < t_{k_0}$. Then construct (u_k, K_k) by induction. If the pair is already constructed at time k, then (u_{k+1}, K_{k+1}) is the solution for the problem

$$\min_{(u,K);K\supseteq K_k;\ u=g(t_{k+1}) \text{ on } \partial\Omega} \left\{ \frac{1}{2}\int_{\Omega\setminus K} \mathbf{A}e(u) : e(u)\, dx + \kappa\mathcal{H}^1(K) \right\}. \tag{9}$$

Then let $\max_k |t_k - t_{k+1}|$ tend to zero and pass to the limit. This should give a time dependent pair $(u(t), K(t))$ which satisfies Griffith's criterium for the evolution of a brittle fracture, which in turn, reduces to write the optimality conditions related to this variational construction. The first mathematical construction in that direction was obtained by Dal Maso and Toader [DMT02] in the simple 2D linearized anti-plane setting, then extended in various directions by other authors [Cha03, DMFT05, FL03, BG14].

But even if the real true object to study is the evolution of $K(t)$ depending on time t, some interesting questions already arise at a freezed time t_0, for which the technics and tools from the original Mumford-Shah problem could be quite useful. A general question of that type is the following : let $K(t_0)$ be a pre-crack at time t_0. How would the crack path grow during the evolution ? Is it continuous in time ? Where it will appear ? And what direction will be privileged ?

Examples of physical quantities related to those questions are the so-called *stress intensity factor* and *energy release rate*, which were the central subject of the papers [CL13, BCL15] and that we present below.

1.4.2 The stationary problem

For $\Omega \subset \mathbb{R}^2$ be open with Lipschitz boundary, we consider the local Griffith energy from fracture mechanics already defined before,

$$\mathcal{G}(u,K) := \frac{1}{2}\int_{\Omega\setminus K} \mathbf{A}e(u) : e(u)\, dx + \kappa\mathcal{H}^1(K), \tag{10}$$

defined on pairs function/set

$$(u,K) \in \mathcal{A}_g(\Omega) := \{K \subset \Omega \text{ is closed and } u \in LD(\Omega\setminus K)\ ;\ u = g \text{ on } \partial\Omega\}.$$

A Griffith minimizer, or minimizer of the Griffith energy, is a solution for the following problem

$$\min_{(u,K)\in\mathcal{A}_g(\Omega)} \mathcal{G}(u,K). \tag{11}$$

We will see later that a minimizer (u,K) indeed do exists, and the singular set K is rectifiable and Alhfors regular.

The main C^1 result that will be described at the end of the this course, is the following partial regularity property for the crack K.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open and sumply connected set with Lipschitz boundary, and let $(u,K) \in \mathcal{A}(\Omega)$ be a local minimizer of the functional $\mathcal{G}$ in $\Omega \subset \mathbb{R}^2$. Then for every isolated connected component $\Gamma \subset K$ there exists $\alpha \in (0,1)$ and an exceptional compact set $Z \subset \Gamma$ such that $\mathcal{H}^1(Z) = 0$ and $\Gamma \setminus Z$ is locally a $\mathcal{C}^{1,\alpha}$ curve.*

1.5 Weak and strong existence for Griffith

Surprisingly, the existence of a Griffith minimizer (i.e. for the problem in (11)) is much harder than the Mumford-Shah problem and was solved only very recently with very nice mathematics. We will not have time to describe in detail all the results but let us just give a few references. Firstly, for the weak existence, Dal Maso defined the famous GSBD space in (2013) [Gia13] and proved the weak existence when the extra term of the form $\int_\Omega |u-g|^2$ is added in the functional. This term does not have a true physical meaning according to crack propagation but is helpful in getting compactness results. More recently, the weak existence without this term in the same space GSBD has been obtained by a very nice paper by Chambolle and Cristmal in [AC19], solving the problem of weak existence in full generality.

According now to the strong existence (together with Ahlfors regularity), it has been also recently obtained by Conti, Focardi and Iurlano in [SMF18] in dimension 2, and then by Chambolle, Conti and Iurlano in [ASF16] for higher dimensions

One key ingredient in all the above results, is the new Poincaré-Korn inequality "outside a small jump set" that was proved by Chambolle, Conti and Francfort in [ASG17]. It is a new and more elementary approach generalizing an inequality which is known to be proved in SBV using the co-area formula (which is not available in SBD).

1.6 Korn Inequalities

A fundamental tool in elasticity theory is the so-called Korn inequality. Let us present this important inequality in this first lecture.

Notation: For $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ we denote by $e(u) := \frac{Du+Du^T}{2}$ the symmetrized gradient.

A "rigid movement" is an affine function $a(x) = Ax + b$ with $A \in \mathcal{M}_n(\mathbb{R})$ a skew-symmetric matrix. In other words $e(a) = 0$. We denote by $LD(\Omega)$ the space of all $u \in L^2(\Omega; \mathbb{R}^n)$ such that $e(u) \in L^2(\Omega)$.

Proposition 1 (Korn inequalities)**.** *Let $\Omega \subset \mathbb{R}^n$ be a smooth domain. There exists $C > 0$ such that:*

1. *For all $u \in LD(\Omega)$ there exists a rigid movement a depending on u satisfying*

$$\int_\Omega |u-a|^2 \, dx \leq C \int_\Omega |e(u)|^2 dx$$

2. *For all $u \in LD(\mathbb{R}^n)$ we have*

$$\int_\Omega |u|^2 \, dx \leq C \int_\Omega |e(u)|^2 \, dx$$

3. *For all $u \in LD(\Omega)$ there exists a skew-symetric matrix A depending on u satisfying*

$$\int_\Omega |Du-A|^2 \, dx \leq C \int_\Omega |e(u)|^2 dx$$

Remark 1.

1. *Inequality 3. is called "Korn" inequality whereas 1. are 2. are called "Poincaré-Korn" inequality.*
2. *The proof of 1. and 2. can be proved "by hand" whereas the proof of 3. is more subtil and uses some harmonic analysis estimates and functional analysis.*

3. 1. 2. and 3. are still true in what is called "John Domains", which is a mild regularity assumption on the domain. But it is not true in general without any regularity assumption on the domain.

4. Inequality 3. implies inequality 2. by the standard Poincaré inequlaity.

5. All the above have an L^p analogue with $p \neq 2$.

Proof. Let us give a sketch of proof for 1. and 2. Actually, one can prove it "by hand" using the fundamental theorem of calculus. The main point is that, if $\xi \in \mathbb{S}^1$ is any direction then

$$\frac{\partial}{\partial \xi}(\xi \cdot u(x)) = \xi \cdot e(u)(x)\xi$$

so that

$$\xi \cdot u(x) - \xi \cdot u(x+\xi) = \int_0^1 \xi \cdot e(u)(x+t\xi)\xi \, dt.$$

Let now $Q = [0,1]^n$ and assume that $u \in C_c^\infty(Q;\mathbb{R}^2)$, in particular, $u = 0$ on ∂Q. Then $\forall 1 \leq i \leq n$

$$e_i \cdot u(x) = e_i \cdot (u(x) - u(x+e_i)) = \int_{[x,x+e_i]} e_i \cdot e(u) \, d\mathcal{H}^1,$$

which implies

$$|e_i \cdot u(x)| \leq \int_{[x,x+e_i]} |e(u)| d\mathcal{H}^1.$$

summing over i and using the equivalence between $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbb{R}^2$ we deduce

$$|u(x)| \leq C \sum_{i=1}^n \int_{[x,x+e_i]} |e(u)| \, d\mathcal{H}^1$$

and then

$$|u(x)|^2 \leq C \sum_{i=1}^n \int_{[x,x+e_i]} |e(u)|^2 \, d\mathcal{H}^1.$$

Finally it is an easy exercice using Fubini's theorem to prove that

$$\int_Q \sum_{i=1}^n \int_{[x,x+e_i]} |e(u)|^2 \, d\mathcal{H}^1 dx \leq \int_Q |e(u)|^2 \, dx,$$

thus integrating the above over x we have proved

$$\int_Q |u(x)|^2 dx \leq C \int_Q |e(u)|^2 \, d\mathcal{H}^1.$$

By doing the same with any cube $Q = [-R,R]^n$ large as we want we obtain the same inequality for any $u \in C_c^\infty(\mathbb{R}^n)$ which finishes the proof of inequality 2.

The proof of 1. is more complicated but can also be done "by hand" from the fundamental theorem of calculus. First we observe that it is enough to find an affine function b (not necessarily rigid) satisfying the same inequality (i.e. b in place of a), because then we can project u on affine functions: the projection a will be rigid and the integral with a will be lower than the one with b.

Then one has to choose an appropriate simplex $T \subset \Omega$ and use an affine approximation of u, denoted by b on this simplex. Next, denoting by y_i the vertices of the simplex T, for each point x in Ω, we can estimate the oscillations of $u \cdot \xi(x) - u \cdot \xi(y_i)$ on each segment $[y_i, x]$, where $\xi_i(x) = \frac{x-y_i}{|x-y_i|}$. Then an elementary argument from convex analysis gives that all the directions $\xi_i(x)$, for $1 \leq i \leq n$ and x fixed are "uniformly non-colinear" so that the norm $|u-b|(x)$ can be estimated by the quantities $|(u-b)\cdot\xi_i|$ and we conclude by integrating everything. For more details, one can read the paper [Chambolle Conti Francfort] where this construction has been performed in the more general case when Ω contains a small singular set K (and therefore one has to furthermore guarantee the segments $[y_i, x]$ to not touch the singular set K. This can be done by averaging everything provided the total length of K is small enough).

Now we come to the proof of inequality 3., which is more difficult and cannot be proved by elementary inequalities as for 1. and 2. The main ingredient for 3., as explained in [Villani] is a nice second order computation. Indeed, for u smooth we have the identity:

$$\frac{\partial^2 u_k}{\partial x_i \partial x_j} = \partial_i (e(u))_{jk} + \partial_j (e(u))_{ik} - \partial_k (e(u))_{ij}. \tag{12}$$

In other words: "$D^2 u$ is a matrix combination of $De(u)$".

Then we can use the following Lemma from harmonic analysis:

Lemma 2. *For any $\Omega \subset \mathbb{R}^n$, smooth, there exists a constant $C > 0$ such that for all $f \in L^2(\Omega)$,*

$$\|\nabla f\|^2_{H^{-1}(\Omega)} \leq n\|f - m_f\|^2_{L^2(\Omega)} \leq C(\Omega)\|\nabla f\|^2_{H^{-1}(\Omega)},$$

where m_f is the average of f on Ω.

Then one can obtain last Korn's inequality by use of this lemma together with the computation in (12). For more details one can see [Villani]. One can also see [finlandais] for a proof in more general John domains. $\square$

2 Lecture Day $\sharp 2$: Regularity for connected almost minimal sets in $\mathbb{R}^2$.

In order to understand the scheme of proof of C^1 regularity for Mumford-Shah type problems, we first review a classical C^1 regularity proof for almost minimizers of length, i.e. when we locally minimize only the surface area term, without the Dirichlet energy.

2.1 Geometrical facts

Definition 1 (Hausdorff distance)**.** *For $K, K' \subset \mathbb{R}^2$ two compact sets we define*

$$d_H(K,K') := \max\left\{ \sup_{x\in K'} dist(x,K) \quad , \quad \sup_{x\in K} dist(x,K') \right\}$$

Proposition 2 (Blashke principle)**.** *If $A \subset \mathbb{R}^2$ is compact, and $(K_n)_{n\in\mathbb{N}}$ is a sequence of compact subsets of A, then we can find a compact subset $K \subset A$ and a subsequence such that $K_{n_k} \to K$ for d_H. Moreover, if K_n is connected for all n then K is connected as well.*

Definition 2. *For any subset $K \subset \mathbb{R}^2$, closed we define*

$$\mathcal{H}^1(K) := \sup_{\varepsilon>0} \inf \left\{ \sum_{i=1}^{\infty} \frac{1}{2}\mathrm{diam}(A_i) \right\},$$

where the infimum is taken over all countable families $\{A_i\}_{i=1}^{\infty}$ of closed sets A_i such that $K \subset \bigcup_{i=1}^{\infty} A_i$ and $\mathrm{diam}(A_i) \leq \varepsilon$.

Here are the things you might need to know about the Hausdorff measure $\mathcal{H}^1$.

Proposition 3.

1. *$K \mapsto \mathcal{H}^1(K)$ is a measure on the Borelians of $\mathbb{R}^2$.*
2. *The measure $\mathcal{H}^1$ generalizes the "lenght", in the sense that if $\gamma : [0,1] \to \mathbb{R}^2$ is a C^1 and injective curve then*

$$\mathcal{H}^1(\gamma([0,1])) = \int_{[0,1]} |\gamma'(t)|dt.$$

3. *We can use $\mathcal{H}^1$ to integrate by parts on smooth domains, for instance on balls, for any smooth vector field $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$*

$$\int_{B(x,r)} \mathrm{div}\, \Phi \, dx = \int_{\partial B(x,r)} \Phi \cdot \nu \, d\mathcal{H}^1,$$

ν being the outer normal vector on $\partial B(0,r)$.

4. *(Theorem of Golab) If $(K_n)_{n\in\mathbb{N}}$ and K are compact and connected sets such that $K_n \to K$ for the Hausdorff distance, then*

$$\mathcal{H}^1(K) \leq \liminf \mathcal{H}^1(K_n).$$

For any closed set $K \subset \mathbb{R}^2$ we define the "flatness"

$$\beta_K(x,r) := \inf_{L \ni x} d_H(K \cap \overline{B}(x,r), L \cap \overline{B}(x,r))$$

where the infimum is taken among all affine lines $L \subset \mathbb{R}^2$ containing x. We will sometimes use the notation $\beta(x,r)$ instead of $\beta_K(x,r)$.

Lemma 3. *Let $K \subset \mathbb{R}^2$ be a closed set containing the origin and satifying, for some constants $C, r_0, \alpha > 0$,*

$$\beta_K(x,r) \leq Cr^\alpha \qquad \forall x \in K \cap B(0,1) \text{ and } r \leq r_0.$$

Then there exists some $a \in (0,1)$ depending only on C, r_0, and α such that $K \cap B(0,a)$ is a 10^{-2}-Lipschitz graph as well as a $\mathcal{C}^{1,\alpha}$ regular curve.

Proof. For every $x \in K \cap B(0,1)$ and $0 < r \leq r_0$ we denote as usual by $L(x,r)$ an affine line which approximates $K \cap B(x,r)$, i.e. such that

$$\max\Big\{ \sup_{z \in K \cap \overline{B}(x,r)} \operatorname{dist}(z, L(x,r)), \sup_{z \in L(x,r) \cap \overline{B}(x,r)} \operatorname{dist}(z,K) \Big\} \leq \beta(x,r) r \leq C r^{1+\alpha}. \tag{13}$$

In addition, we denote by $\tau(x,r) \in \mathbb{S}^1/\{\pm 1\}$ a non oriented unit vector which is tangent to $L(x,r)$ and defined modulo ± 1. For $\tau_1, \tau_2 \in \mathbb{S}^1/\{\pm 1\}$ we can use for instance the complete distance

$$d_S(\tau_1, \tau_2) := \min(|\tau_1 - \tau_2|, |\tau_1 + \tau_2|).$$

We will prove the Lemma within 4 steps.

Step 1. Existence of tangents. For all $k \in \mathbb{N}$ we denote by $r_k := 2^{-k} r_0$. We claim that $\tau(x, r_k)$ converges to some vector $\tau(x)$ at x when k goes to $+\infty$. For that purpose we first prove that for all $k \geq 0$, and for all $x \in K \cap B(0,1)$ we have

$$d_S\big(\tau(x, r_{k+1}), \tau(x, r_k)\big) \leq 9 C r_k^\alpha.$$

Indeed, let $z := x + \tau(x, r_{k+1}) r_{k+1} \in L(x, r_{k+1})$. Because of (13) we know that there exists $y \in K \cap \overline{B}(x, r_{k+1})$ such that $|z - y| \leq C r_{k+1}^{1+\alpha}$ and in particular,

$$r_{k+1} - C r_{k+1}^{1+\alpha} \leq |y - x| \leq r_{k+1} \tag{14}$$

Then, if we denote by $v := \frac{y-x}{|y-x|}$ we have that

$$\begin{aligned} d_S\big(v, \tau(x, r_{k+1})\big) \leq |v - \tau(x, r_{k+1})| &= \Big|\frac{y-x}{|y-x|} - \frac{z-x}{r_{k+1}}\Big| \\ &\leq \Big|\frac{y-x}{|y-x|} - \frac{y-x}{r_{k+1}}\Big| + \frac{1}{r_{k+1}}|z-y| \\ &\leq \frac{|r_{k+1} - |y-x||}{r_{k+1}} + C r_{k+1}^\alpha \\ &\leq 2 C r_{k+1}^\alpha, \end{aligned} \tag{15}$$

where we also have used (14) to get the last inequality.

But then similarly, since $y \in B(x, r_k)$, there exists $z' \in L(x, r_k)$ such that $|y - z'| \leq C r_k^{1+\alpha}$. By (14) again we can estimate

$$|z' - x| \leq |y - x| + |z' - y| \leq r_{k+1} + C r_k^{1+\alpha}$$

and

$$|z' - x| \geq |y - x| - |z' - y| \geq r_{k+1} - C r_{k+1}^{1+\alpha} - C r_k^{1+\alpha} \geq r_{k+1} - 2C r_k^{1+\alpha},$$

thus a computation similar to the one of (15) leads to

$$\begin{aligned}
d_S\big(v, \tau(x, r_k)\big) \leq |v - \frac{z' - x}{|z' - x|}| &= |\frac{y - x}{|y - x|} - \frac{z' - x}{r_{k+1}}| \\
&\leq |\frac{y - x}{|y - x|} - \frac{y - x}{r_{k+1}}| + |\frac{y - x}{r_{k+1}} - \frac{z' - x}{r_{k+1}}| + |\frac{z' - x}{|z' - x|} - \frac{z' - x}{r_{k+1}}|. \\
&\leq C r_{k+1}^{\alpha} + C \frac{r_k^{1+\alpha}}{r_{k+1}} + 2C \frac{r_k^{1+\alpha}}{r_{k+1}} \leq 7C r_k^{\alpha}. \qquad (16)
\end{aligned}$$

Gathering together the above two inequalities we obtain

$$d_S(\tau(x, r_k), \tau(x, r_{k+1})) \leq d_S(\tau(x, r_k), v) + d_S(v, \tau(x, r_{k+1})) \leq 9C r_k^{\alpha} = 9C r_0^{\alpha} 2^{-k\alpha},$$

as claimed. It follows that for all $k, l \geq k_0$,

$$d_S\big(\tau(x, r_k), \tau(x, r_l)\big) \leq \sum_{i=k_0}^{+\infty} 9C r_0^{\alpha} 2^{-i\alpha} = 2^{-k_0\alpha} \left(\frac{9C r_0^{\alpha}}{1 - 2^{-\alpha}}\right).$$

Since the latter can be made as small as desired, provided k_0 is big enough, we deduce that $\tau(x, r_k)$ is a Cauchy sequence in $\mathbb{S}^1/\{\pm 1\}$. Therefore it converges to some vector, that we denote $\tau(x)$, for all $x \in K \cap B(0, 1)$. In particular, letting $l \to +\infty$ we get the following estimate, for all $k \geq 0$,

$$d_S\big(\tau(x, r_k), \tau(x)\big) \leq C' r_k^{\alpha},$$

where

$$C' := \left(\frac{9C}{1 - 2^{-\alpha}}\right).$$

Moreover, it can be easily seen through the distance estimate (13), that $x + \mathbb{R}\tau(x)$ is a tangent line for the set K at point x.

Step 2. Hölder estimate on tangents. We now prove that the mapping $x \mapsto \tau(x)$ is Hölder continuous. Let x and y be two different points of $K \cap B(0, 1)$ and let $\rho := |y - x|$. Assume first that $\rho \leq r_0/4$ and let $k \in \mathbb{N}$ be such that

$$r_{k+2} \leq \rho \leq r_{k+1}.$$

We have that

$$\begin{aligned}
d_S\big(\tau(x), \tau(y)\big) &\leq d_S\big(\tau(x), \tau(x, r_k)\big) + d_S\big(\tau(x, r_k), \tau(y, r_k)\big) + d_S\big(\tau(y, r_k), \tau(y)\big) \\
&\leq 2C' r_k^{\alpha} + d_S\big(\tau(x, r_k), \tau(y, r_k)\big). \qquad (17)
\end{aligned}$$

Now to estimate $d_S\big(\tau(x,r_k),\tau(y,r_k)\big)$, we notice that $y \in B(x,r_k)$ thus there exists $z \in P(x,r_k)$ such that $|y-z| \leq Cr_k^{1+\alpha}$. Let us denote by $v := \frac{y-x}{y-x}$. By a computation very similar to the one of (15) or (16) above, we get

$$d_S(v,\tau(x,r_k)) \leq 2Cr_k^\alpha.$$

Inverting the role of x and y yields also

$$d_S(v,\tau(y,r_k)) \leq 2Cr_k^\alpha,$$

from which we deduce, returning back to (17), that

$$d_S\big(\tau(x),\tau(y)\big) \leq 2C'r_k^\alpha + 2Cr_k^\alpha \leq 2(C'+C)2^{2\alpha}r_{k+2}^\alpha \leq 2(C'+C)2^{2\alpha}|x-y|^\alpha. \tag{18}$$

Now in the case when $\rho \geq r_0/4$ we can simply estimate

$$d_S\big(\tau(x),\tau(y)\big) \leq 2 \leq 2\frac{4^\alpha}{r_0^\alpha}|x-y|^\alpha,$$

which finally yields, for general $x,y \in K \cap B(0,1)$,

$$d_S\big(\tau(x),\tau(y)\big) \leq C''|x-y|^\alpha, \tag{19}$$

with $C'' := \max\big(24^\alpha r_0^{-\alpha}, 2(C'+C)2^{2\alpha}\big)$.

In other words, we have proved that K admits a tangent everywhere on $B(0,1)$ and that tangent lines behaves nicely. We will prove now that $K \cap B(0,a)$ is a curve for a small enough. Actually, a convenient way to prove that fact is to show the following stronger property: for some $a \in (0,1)$ small enough, $K \cap B(0,a)$ is a Lipschitz graph.

Step 3. $K \cap B(0,a)$ is a Lipschitz graph. Let a be a small parameter that will be fixed later. We first show that for a small enough, $K \cap B(0,a)$ is a graph above the line $\mathbb{R}\tau(0)$, that we assume for simplicity that it is oriented by e_1. Notice that for all $x \in K \cap B(0,a)$,

$$d_S(\tau(x),e_1) \leq C''(2a)^\alpha, \tag{20}$$

which means that for a small, all the tangents are oriented almost horizontally in $K \cap B(0,a)$.

Now we assume by contradiction that one can find two points $x,y \in K \cap B(0,a)$ such that $x_1 = y_1$. Let $\rho := 10|x-y| = 10|x_2-y_2| \leq 20a \leq r_0/10$, and let k be such that

$$r_{k+1} \leq \rho \leq r_k.$$

By (20) we know that $d_S(\tau(x),\tau(y)) \leq 2C''(2a)^\alpha$. Let us denote $T_x := x + \mathbb{R}\tau(x)$. Since $y \in B(x,r_k)$, by (13) we infer that

$$\mathrm{dist}(y,T_x) \leq \mathrm{dist}(y,L(x,r_k)) \leq Cr_k^{1+\alpha} \leq Ca^\alpha r_k.$$

We deduce that, for some universal constant $c_0 > 0$,

$$|x_2-y_2| = \mathrm{dist}(y,x+\mathbb{R}e_1) \leq \mathrm{dist}(y,T_x)+c_0r_kd_S(\tau_x,e_1) \leq a^\alpha(C+c_0C''2^\alpha)r_k \leq a^\alpha C'''|x_2-y_2|,$$

which is a contradiction for a small enough (depending on C'''), which proves that $K \cap B(0,a)$ must be a graph above the segment $[-a,a] \times \{0\}$. Now to prove that the graph is 10^{-3}-Lipschitz for a slightly smaller enough, we can reproduce the same argument but for $x, y \in K \cap B(0,a)$ satisfying now, by contradiction, $|x_2 - y_2| \leq 10^{-3}|x_1 - y_1|$. The details are left to the reader.

Step 4. Conclusion. We have proved that $K \cap B(0,a)$ is the 10^{-3}-Lipchitz graph of some function f on $[-a,a]$. Moreover, the tangent line to the graph of f at point $(t, f(t))$, which exists for a.e. $t \in [-a,a]$, concides with the tangent line $x + \mathbb{R}\tau(x)$ to K at point $x = (t, f(t))$. Since the map $x \mapsto \tau(x)$ is α-Hölder continuous, it follows that the map $t \mapsto f'(t)$ coïcides a.e. on $[-a,a]$ with an α-Hölder continuous function. A simple smoothing argument then implies that $f \in \mathcal{C}^{1,\alpha}$ on $[-a,a]$, and $K \cap B(0,a)$ is then a $\mathcal{C}^{1,\alpha}$ curve. □

Lemma 4. *Let $K \subset \mathbb{R}^2$ be a $\mathcal{H}^1$-rectifiable set. Then for all $0 < s < r$ and $x \in \mathbb{R}^2$ we have*

$$\int_s^r \#(K \cap \partial B(x_0, t))\, dt \leq \mathcal{H}^1(K \cap B(x_0, r) \setminus B(x_0, s)). \tag{21}$$

Proof. Applying the area formula [AFP00, Theorem 2.91] to the $\mathcal{H}^1$-rectifiable set $E := K \cap B(x_0, r) \setminus B(x, s)$ and the Lipschitz function $f : x \mapsto |x|$ yields

$$\int_s^r \#(K \cap \partial B(x_0, t))\, dt = \int_{\mathbb{R}} \mathcal{H}^0(E \cap f^{-1}(t))\, dt = \int_E \mathbf{J}d^E f \, d\mathcal{H}^1,$$

where, $\mathcal{H}^1$-a.e. in E, $\mathbf{J}d^E f$ denotes the 1-dimensional coarea factor associated to the tangential differential df^E. Since E admits an approximate tangent line oriented by a unit vector τ at $\mathcal{H}^1$-a.e. points, we deduce that

$$\mathbf{J}d^E f_x = \left| \frac{x}{|x|} \cdot \tau \right| \leq 1 \quad \mathcal{H}^1\text{-a.e. in } E,$$

which leads to (21). □

2.2 Regularity for connected almost minimal sets in $\mathbb{R}^2$.

For $\Omega \subset \mathbb{R}^2$, bounded, we denote by $\mathcal{K}(\Omega)$ all the compact connected sets $K \subset \overline{\Omega}$. We denote by $\mathcal{H}^1$ the 1-dimensional Hausdorff measure. A gauge function h is an increasing function $h : \mathbb{R}^+ \to \mathbb{R}^+$.

Definition 3. *We say that $K \in \mathcal{K}(\Omega)$ is almost minimal in Ω with jauge function h if, for all ball $B \subset \Omega$ of radius $r > 0$ and competitor $K' \in \mathcal{K}(\Omega)$ for K in B (which means $K = K'$ in $\Omega \setminus B$) we have:*

$$\mathcal{H}^1(K \cap B) \leq \mathcal{H}^1(K' \cap B) + rh(r).$$

Example 1. *For instance it can be proved by a variant Golab's theorem (see [DPLM17, Theorem 3.6]) that the solution of the following weighted Steiner minimizing problem do exists: for some given points $(x_i)_{i=1}^N \subset \Omega$, and for $w : \mathbb{R}^2 \to \mathbb{R}$ continuous, and bounded from below (i.e. $\inf_x w(x) \geq \alpha > 0$),*

$$\min_{K \in \mathcal{K}(\Omega)} \int_K w(x)\, d\mathcal{H}^1(x),$$

and a minimizer is an almost minimal set.

Remark 2. *From the definition of almost minimality, one easily see by taking a sequence of open balls $B(x, r+\varepsilon)$ with $\varepsilon \to 0$, that if the competitor $K' \in \mathcal{K}(\Omega)$ satisfies $K = K'$ in $\Omega \setminus \overline{B}$ we have in this case:*

$$\mathcal{H}^1(K \cap \overline{B}) \leq \mathcal{H}^1(K' \cap \overline{B}) + rh(r).$$

Remark 3 (Alhfors-regularity)**.** *If $K \in \mathcal{K}(\Omega)$ is almost minimal in Ω with jauge function h then for every $r < r_0 := \mathrm{diam}(K)$ such that $B(x,r) \subset \Omega$, by considering the competitor $K' := (K \setminus B(x,r)) \cup \partial B(x,r)$ we obtain the upper bound*

$$\mathcal{H}^1(K \cap B(x,r)) \leq 2\pi r + h(r)r \leq (2\pi + h(r_0))r.$$

On the other hand since K is connected and $r < r_0 := \mathrm{diam}(K)$ we also have the lower bound $\mathcal{H}^1(K \cap B(x,r)) \geq r$.

Remark 4 (Uniform rectifiability)**.** *K is Ahlfors-regular. Since K is furthermore connected with locally finite $\mathcal{H}^1$ measure, it follows that K is automatically uniformly rectifiable in the sense of David and Semmes.*

The main purpose of this section is to prove the following ε-regularity result.

Theorem 2.1 (Flatness implies regularity)**.** *Let $K \in \mathcal{K}(\Omega)$ be an almost minimal set in Ω with jauge function h satisfying $h(r) \leq Cr^\alpha$ for all $r > 0$. Then*

$$\beta(x,r) + h(r) \leq 10^{-3} \Longrightarrow K \cap B(x, r/10) \text{ is a } C^{1,\alpha/2} \text{ regular curve.} \tag{22}$$

Actually, the above result is not the better regularity result that one can obtain about almost minimal sets. Here is the optimal statement:

Theorem 2.2. *Let $K \in \mathcal{K}(\Omega)$ is almost minimal in Ω with jauge function h satisfying $h(r) \leq Cr^\alpha$. Then K is a finite union of $C^{1,\alpha/2}$ curves meeting only by 3 with 120 angles.*

But in those notes we will mainly focus only on the simpler Theorem 2.1 that we shall try to adapt for other problems later, for which the better Theorem 2.2 is not available. Notice however the interesting following corollary of Theorem 2.1.

Remark 5. *Using some "porosity" type arguments and Carlson measure estimates, if* (22) *holds for some uniformly rectifiable set K, then K is $C^{1,\alpha}$ regular outside a singular set of dimension $s < 1$.*

In the sequel we give a complete proof of Theorem 2.1.

2.3 The height estimate

Lemma 5. *Let $\gamma : [0,1] \to \mathbb{R}^2$ be a curve with endpoints $z = \gamma(0)$ and $z' = \gamma(1)$, with image $\Gamma := \gamma([0,1])$. Then*

$$\mathrm{dist}(y, [z,z'])^2 \leq \frac{\mathcal{H}^1(\Gamma)\big(\mathcal{H}^1(\Gamma) - |z'-z|\big)}{2} \quad \text{for all } y \in \Gamma. \tag{23}$$

Proof. Let $\bar{y}$ be a maximizer of the function $y \in \Gamma \mapsto \mathrm{dist}(y, [z, z'])$, i.e., $\bar{y}$ is the most distant point in Γ to the segment $[z, z']$, and define $d =: \mathrm{dist}(\bar{y}, [z, z'])$. Let us consider the point $y' \in \mathbb{R}^2$ making (z, z', y') an isosceles triangle with same height d. Denoting by $a := |z - z'|/2$ and $L := |y' - z|$, according to Pythagoras Theorem, we have

$$d^2 = L^2 - a^2 = (L - a)(L + a).$$

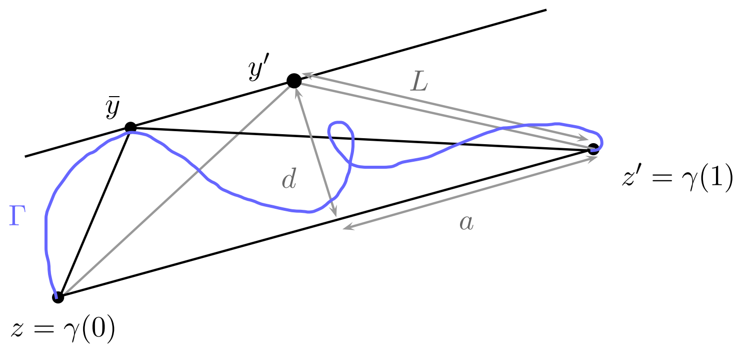

Figure 1: The height estimate from Pythagoras inequality.

Thus $\mathcal{H}^1(\Gamma) \geq |z - \bar{y}| + |\bar{y} - z'| \geq 2L$ and $\mathcal{H}^1(\Gamma) \geq |z - z'|$ so that

$$d^2 \leq \frac{1}{4} \left(\mathcal{H}^1(\Gamma) - |z - z'|\right) \left(\mathcal{H}^1(\Gamma) + |z - z'|\right) \leq \frac{\mathcal{H}^1(\Gamma)\big(\mathcal{H}^1(\Gamma) - |z - z'|\big)}{2},$$

which proves (23). $\square$

Corollary 1 (Control of flatness by the Excess)**.** *Let $K \in \mathcal{K}(\Omega)$ be Ahlfors-regular. Let $x \in K$ and $r > 0$ be such that $K \cap \overline{B}(x, r)$ is connected and $\sharp(K \cap \partial B(x, r)) = 2$. Then denoting by $\{z, z'\}$ those two points, if they lie "on both sides" we have*

$$\beta_K(x, r)^2 \leq \frac{C}{r} \left(\mathcal{H}^1(K \cap B(x, r)) - |z - z'|\right).$$

2.4 Proof of Theorem 2.1

We start with an easy density bound.

Proposition 4. *Let $K \in \mathcal{K}(\Omega)$ be an almost minimal set with gauge function h and such that $\beta(x, r) \leq \frac{1}{10}$. Then*

$$\mathcal{H}^1(K \cap B(x, r)) \leq 2r + 10r\beta(x, r) + rh(r).$$

Proof. We denote by $L(x, r)$ the best line that realizes the infimum in the definition of $\beta(x, r)$ and

$$W := \{x \in \partial B(x, r) \ : \ dist(x, L(x, r)) \leq r\beta(x, r)\}.$$

We notice that

$$\mathcal{H}^1(W) \leq 4r \arcsin(\beta(x, r)) \leq 10r\beta(x, r),$$

because $\beta(x,r) \leq 1/10$ and $\arcsin'(t) \leq 2$ for $t \in [0,1/10]$. We then use the competitor defined by

$$K' := (K \setminus B(x,r)) \cup (L(x,r) \cap B(x,r)) \cup W.$$

It is easy to check that $K' \in \mathcal{K}(\Omega)$ and by almost minimality one has

$$\mathcal{H}^1(K \cap B(x,r)) \leq \mathcal{H}^1(K' \cap B(x,r)) + rh(r) = 2r + 10r\beta(x,r) + rh(r),$$

and the proposition is proved. □

A topological Lemma (left to the reader).

Lemma 6. *Let $K \in \mathcal{K}(\Omega)$. If $\sharp(K \cap \partial B(x,r)) = 2$ and $K \cap B(x,r)$ is not connected, then $K \setminus B(x,r)$ is connected.*

Corollary 2. *Let $K \in \mathcal{K}(\Omega)$ be an almost minimal set with gauge function h. Assume that $h(r_0) \leq 1/2$. Then for every $r \leq \min(r_0, diam(K))$ and $x \in K$ such that $B(x,r) \subset \Omega$, the following holds:*

1. *$\sharp(\partial B(x,r)) \geq 2$*
2. *if $\sharp(\partial B(x,r)) = 2$ then $K \cap B(x,r)$ is connected.*
3. *if $\sharp(\partial B(x,r)) = 2$ and $\beta(x,r) \leq 1/2$ then $K \cap \partial B(x,r)$ are on "both sides"*

Proof. If $\sharp(K \cap \partial B_r) = 0$ or $\sharp(K \cap \partial B_r) = 1$ then $K \setminus B(x,r)$ is connected thus $K \setminus B(x,r)$ is a competitor. But since $x \in K$ and $r < diam(K)$, it follows that $\mathcal{H}^1(K \cap B(x,r)) \geq r$. On the other hand it is an almost minimizer thus

$$r \leq \mathcal{H}^1(K \cap B(x,r)) \leq rh(r) \leq \frac{1}{2}r,$$

a contradiction. This achieves the proof of (1).

To prove (2), we assume by contradiction that $K \cap B(x,r)$ is not connected. Then Lemma 6 says that $K \setminus B(x,r)$ is a competitor thus again, we have a contradiction arguing exactly as for the proof of (1).

To prove the last item we argue by contradiction. If $K \cap \partial B(x,r)$ lie not "on both sides" then the competitor given by a little wall on one side gives a contradiction. □

We can now give the main flatness estimate from which the proof of Theorem 2.1 easily follows using also Lemma 3.

Proposition 5. *Let $K \in \mathcal{K}(\Omega)$ be an almost minimal set with gauge function h. If*

$$\beta(x,r) + h(r) \leq 10^{-3},$$

then

$$\beta(x, \frac{r}{2}) \leq C\sqrt{h(r)}.$$

Proof. We start applying Proposition 4 which says that

$$\mathcal{H}^1(K \cap B(x,r)) \leq 2r + 10r\beta(x,r) + rh(r) \leq 2r + 10^{-2}r.$$

This estimate, together with the coarea formula (21) will allow us to find some radius $s \in [\frac{r}{2}, r]$ such that $\sharp K \cap \partial B(x,s) = 2$. Indeed,

$$\int_0^r \#(K \cap \partial B(x_0,t))\, dt \leq \mathcal{H}^1(K \cap B(x_0,r)) \leq 2r + 10^{-2}r.$$

Now let $A_2 \subset (0,r)$ be defined as

$$A_2 := \{t \in (0,r)\,;\, \#(K \cap \partial B(x_0,t)) \neq 2\}.$$

By Corollary 2 we know that $\#(K \cap \partial B(x_0,t)) > 3$ for all $t \in A_2$ thus

$$\int_0^r \#(K \cap \partial B(x_0,t))\, dt \geq 3|A_2| + 2(r - |A_2|) = |A_2| + 2r,$$

and therefore

$$|A_2| \leq 10^{-2}r.$$

This means that there exists $s \in [r/2, 2]$ such that $s \notin A_2$. In particular $\#(K \cap \partial B(x_0,s)) = 2$ and by corollary 2 we know that $K \cap B(x_0,s)$ is connected and $\{z,z'\} := K \cap \partial B(x_0,s)$ lie on "both sides". We can apply Corollary 1 which gives

$$\beta_K(x,s)^2 \leq \frac{C}{s}\left(\mathcal{H}^1(K \cap B(x,s)) - |z - z'|\right).$$

But the competitor $(K \setminus B(x_0,s)) \cup [z,z']$ is connected, thus by minimality we get

$$\mathcal{H}^1(K \cap B(x,s)) \leq |z - z'| + sh(s),$$

so finally

$$\beta_K(x,r/2)^2 \leq 4\beta(x,s)^2 \leq Ch(r).$$

$\square$

3 Lecture day ♯3: Existence and regularity for connected minimizers of the Mumford-Shah functional

Now we focus on the Mumford-Shah functional, for $\Omega \subset \mathbb{R}^2$ a bounded, connected, open set with smooth boundary and $g \in L^\infty(\Omega)$,

$$J(u, K) := \min_{(u,K)} \int_{\Omega \setminus K} |\nabla u|^2 \, dx + \int_\Omega |u - g|^2 \, dx + \mathcal{H}^1(K),$$

where the minimum is taken among all pairs $(u, K) \in \mathcal{A}_C(\Omega)$ and

$$\mathcal{A}_C(\Omega) := \{(u, K) \; ; \; K \subset \overline{\Omega} \text{ is closed, connected, and } u \in W^{1,2}_{loc}(\Omega \setminus K)\}.$$

3.1 Existence

Proposition 6. *For every $g \in L^\infty(\Omega)$, there exists a minmizer for the problem*

$$\min_{(u,K) \in \mathcal{A}_C} J(u, K).$$

Proof. First we notice that the infimum is not $+\infty$ because for instance the pair $(1, \emptyset)$ is admissible and has finite energy.

We let (u_n, K_n) be a minimizing sequence. Since K_n is a sequence of compact and connected sets in the compact set $\overline{\Omega}$, by Blashke we know that (up to extract a subsequence) $K_n \to K \subset \overline{\Omega}$ for the Hausdorff distance where K is compact connected and by Golab we also know that

$$\mathcal{H}^1(K) \le \liminf_{n \to +\infty} \mathcal{H}^1(K_n).$$

Now we use the energy bound

$$\sup_n \int_{\Omega \setminus K_n} |\nabla u_n|^2 \, dx + \int_\Omega |u_n - g|^2 \, dx \le C \tag{24}$$

which in particular says that ∇u_n is uniformly bounded in $L^2(\Omega \setminus K_n) = L^2(\Omega)$. This means that, up to a subsequence, ∇u_n weakly converge in $L^2(\Omega)$ to some vector valued function $\Phi \in L^2(\Omega)$ and due to the lower semicontinuity of the Dirichlet energy under weak convergence (because it is a convex functional) we infer that

$$\int_\Omega |\Phi|^2 \, dx \le \liminf_{n \to +\infty} \int_\Omega |\nabla u_n|^2 \, dx.$$

By the argument on $u_n - g$, we also have that u_n converges weakly in L^2 to some function u and that

$$\int_\Omega |u - g|^2 \, dx \le \liminf_{n \to +\infty} \int_\Omega |u_n - g|^2 \, dx.$$

To finish the proof, it suffice to prove that $u \in W^{1,2}_{loc}(\Omega \setminus K)$ and that $\Phi = \nabla u$ in $\Omega \setminus K$ because then (u, K) would belong to $\mathcal{A}_C(\Omega)$ and the above semicontinuity inequalities proves that it is a minimizer.

For that purpose we let $A_j \subset \Omega \setminus K$ be a countable exhaustion of $\Omega \setminus K$ by smooth domains, i.e. $\Omega \setminus K = \bigcup_j A_j$ and $A_j \subset A_{j+1}$. From the Hausdorff convergence of K_n to K, we know that $A_j \subset \Omega \setminus K_n$ for n large enough, depending on j, thus by (24) we deduce that the sequence u_n is bounded in $W^{1,2}(A_j)$ for each A_j fixed (for n large enough depending on j). Therefore, we can extract a subsequence (depending on j), such that u_n converges strongly in $L^2(A_j)$ and weakly in $W^{1,2}(A_j)$ to some function, which by uniqueness of the limit must be u. We deduce that $\nabla u = \Phi$ in all the A_j, thus a.e. in $\Omega \setminus K$, and $u \in W^{1,2}(A_j)$ for all j thus finally $u \in W^{1,2}_{loc}(\Omega \setminus K)$. $\square$

Let

$$\mathcal{A}_C := \{(u, K) \;:\; u \in W^{1,2}(\Omega \setminus K) \text{ and } K \subset \overline{\Omega} \text{ is closed and connected}\}.$$

In the rest of this lecture we study the following problem, for some given smooth function $g : \mathbb{R}^2 \to \mathbb{R}$,

$$\min_{(u,K)\in\mathcal{A}_C \,:\, u=g \text{ on } \partial\Omega\setminus K} \qquad \int_{\Omega\setminus K} |\nabla u|^2 dx + \mathcal{H}^1(K). \tag{25}$$

3.2 C^1 regularity for Mumford-Shah

We now come to the proof of Bonnet about the C^1 regularity of the singular set, for a Mumford-Shah minimizer.

Theorem 3.1 (Bonnet)**.** *Let (u, K) be a minimizer in $\mathcal{A}_C$. Then K is a finite union of $C^{1,\alpha}$ curves.*

The proof is based on two main ingredients: monotonicity and blow-up technics. We will not prove the entire of Bonnet's result but give some ideas.

One of his key tool is the monotonicity behavior of the Dirichlet energy that will need the following preliminary elementary facts.

3.2.1 Preliminaries

Lemma 7 (Poincaré-Wirtinger)**.** *Let $I \subset \partial B(0, r)$ be an arc of circle and $u \in W^{1,2}(I)$. Then*

$$\int_I |u - m_u|^2 d\mathcal{H}^1 \leq \left(\frac{\mathcal{H}^2(I)}{\pi}\right)^2 \int_I \left|\frac{\partial u}{\partial \tau}\right|^2 d\mathcal{H}^1,$$

where $m_u := \frac{1}{\mathcal{H}^1(I)} \int_I u \, d\mathcal{H}^1$.

Proof. use Fourier Series. $\square$

Definition 4. *Let $K \in \mathcal{K}_C(\Omega)$. We say that $u \in W^{1,2}(\Omega \setminus K)$ is a local Dirichlet energy minimizer (in short, $u \in LDEM(\Omega \setminus K)$) if for all ball $B \subset \Omega$ and for all $v \in W^{1,2}(\Omega \setminus K)$ satisfying $v = u$ in $\Omega \setminus B$ we have*

$$\int_{B\setminus K} |\nabla u|^2 dx \leq \int_{B\setminus K} |\nabla v|^2 dx.$$

Remark 6. *If $u \in LDEM(\Omega \setminus K)$ then it satisfies the following problem in a weak sense,*

$$(P) \begin{cases} -\Delta u = 0 & \text{in } \Omega \setminus K \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } K. \end{cases}$$

Remark 7. *If (u, K) is a Mumford-Shah minimizer (i.e. minimizes (25)), then $u \in LDEM(\Omega \setminus K)$.*

Proposition 7 (Integration by parts). *If $K \in \mathcal{K}_C(\Omega)$ and $u \in LDEM(\Omega \setminus K)$ then for all ball $B(x, r) \subset \Omega$ we have*

$$\int_{B(x,r)\setminus K} |\nabla u|^2 dx = \int_{\partial B(x,r)} u \frac{\partial u}{\partial \nu} d\mathcal{H}^1.$$

Proof. We can get this expression formally by use of an integration by parts and Remark (6), namely,

$$\begin{aligned} \int_{B(x,r)\setminus K} |\nabla u|^2 dx &= -\int_{B(x,r)} u\Delta u + \int_{\partial B(x,r)} u \frac{\partial u}{\partial \nu} d\mathcal{H}^1 + \int_{K\cap B(x,r)} u \frac{\partial u}{\partial \nu} d\mathcal{H}^1 \\ &= \int_{\partial B(x,r)} u \frac{\partial u}{\partial \nu} d\mathcal{H}^1, \end{aligned}$$

where we have use the equation in problem (P). But to prove rigorously this formula we can use the weak formulation of Problem (P), which is

$$\int_{\Omega\setminus K} \nabla u \cdot \nabla \varphi dx = 0$$

for all $\varphi \in W^{1,2}(\Omega)$ compactly supported inside Ω. Let $\varphi_\varepsilon = u\psi_\varepsilon$ where ψ_ε is a cut-off function, which is of the form $\psi_\varepsilon = f_\varepsilon(|x|)$, where $f_\varepsilon = 1$ on $[0, (1-\varepsilon)r]$ and 0 outside $[r+\varepsilon, +\infty[$. Testing with this function φ_ε yields

$$\int_{\Omega\setminus K} \nabla u \cdot u \nabla \psi_\varepsilon dx + \int_{\Omega\setminus K} \nabla u \cdot \psi_\varepsilon \nabla u dx = 0.$$

As $\varepsilon \to 0$, one easily gets that the second term converges to $\int_{B(x,r)\setminus K} |\nabla u|^2 dx$ by the dominated convergence theorem. For the first term, we compute

$$\nabla \psi_\varepsilon = \frac{x}{2\varepsilon|x|} \mathbf{1}_{B(x,r+\varepsilon)\setminus B(x,r-\varepsilon)},$$

from which we get

$$\int_{\Omega\setminus K} \nabla u \cdot u \nabla \psi_\varepsilon dx \to \int_{\partial B(x,r)\setminus K} u \frac{\partial u}{\partial \nu} d\mathcal{H}^1.$$

□

3.2.2 The monotonicity formula of Bonnet

One of our main ingredient to prove Lemma **??** will be a monotonicity formula.

Proposition 8 (Monotonicity Formula of Bonnet)**.** *Let $\Omega \subset \mathbb{R}^2$ be open, and assume that $K \subset \Omega$ is a closed and connected set of finite length. Let u be an energy minimizer i.e. satisfying*

$$\int_{B\setminus K} |\nabla u|^2 dx \le \int_{B\setminus K} |\nabla v|^2 dx,$$

for any $B \subset \Omega$ and for any v that is equal to u in $\Omega \setminus B$ (the function u is therefore the weak solution of a Neumann problem, $\Delta u = 0$ in $\Omega \setminus K$ and $\frac{\partial u}{\partial \nu} = 0$ on K).

For any point $x_0 \in K$ we denote

$$E(r) := \int_{B(x_0,r)\setminus K} |\nabla u|^2 dx.$$

Then $r \mapsto E(r)/r$ is an increasing function of r on $(0, dist(x_0, \partial\Omega))$. As a consequence, the limit $\lim\limits_{r\to 0} (E(r)/r)$ exists and is finite.

Moreover, if $r \mapsto E(r)/r$ is a non zero constant on some interval (a,b), then for $r \in (a,b)$, $K \cap \partial B_r$ is a single point and the restriction of u on $\partial B_r \setminus K$ for $r \in (a,b)$ must be the optimal function in Wirtinger's inequality.

Proof. Let us assume without loss of generality that x_0 is the origin. Firstly, it is easy to show that E admits a derivative a.e. and

$$E'(r) := \int_{\partial B(0,r)\setminus K} |\nabla u|^2 dx. \tag{26}$$

In addition E is absolutely continuous. Therefore, to prove the monotonicity of $r \mapsto \frac{1}{r}E(r)$, it is enough to prove the inequality

$$E(r) \le rE'(r) \qquad \text{for a.e. } r \le r_0, \tag{27}$$

because this implies $\left(\frac{1}{r}E(r)\right)' \ge 0$ a.e.

We will need Wirtinger's inequality (see e.g. page 301 of [Dav05]), i.e. for any arc of circle $I_r \subset \partial B(0,r)$ and for $g \in W^{1,2}(I_r)$ we have

$$\int_{I_r} |g - m_g|^2 dK \le 4\left(\frac{|I_r|}{2\pi}\right)^2 \int_{I_r} |g'|^2 dK \tag{28}$$

where m_g is the average of g on I_r. The constant 4 here is optimal, and is achieved by the function $\sin(\theta/2)$ on the arc of circle $\{\theta \in]-\pi,\pi[\}$. This will be needed later.

Observe that since K has a finite length, we know that $\sharp K \cap \partial B(0,r)$ is finite for a.e. $r \in (0, r_0)$. We take such a radius and decompose $S_r := \partial B(0,r) \setminus K$ into a finite number of arcs of circle denoted I_j, for $j = 1..N$. Moreover since K is closed and connected, for each j there exists a geodesic simple curve $F_j \subset K$ connecting the two endpoints of I_j (here is where connectedness plays a role). We denote by D_j the domain delimited by I_j and F_j. Observe that the domains D_j for $j = 1..N$ are disjoint.

The Gauss-Green formula (that can by justified easily here by a variational argument) applied in $B(0,r)$ yields

$$\int_{B(0,r)\setminus K} |\nabla u|^2 dx = \sum_{i=1}^{N} \int_{I_j} u\frac{\partial u}{\partial \nu} dK, \tag{29}$$

and applied in D_j gives

$$\int_{I_j} \frac{\partial u}{\partial \nu} dx = \int_{D_j} \Delta u dx = 0.$$

Denoting by m_j the average of u on I_j we deduce that

$$\int_{I_j} u \frac{\partial u}{\partial \nu} d\sigma = \int_{I_j} (u - m_j) \frac{\partial u}{\partial \nu} d\sigma. \tag{30}$$

Then by use of Cauchy-Schwarz inequality and $ab \leq \frac{1}{2\lambda} a^2 + \frac{\lambda}{2} b^2$ we can write

$$\begin{aligned}
\int_{I_j} |u - m_j| \left| \frac{\partial u}{\partial \nu} \right| d\sigma &\leq \left(\int_{I_j} |u - m_j|^2 \right)^{\frac{1}{2}} \left(\int_{I_j} |\frac{\partial u}{\partial \nu}|^2 \right)^{\frac{1}{2}} \\
&\leq \frac{1}{2\lambda} \int_{I_j} |u - m_j|^2 + \frac{\lambda}{2} \int_{I_j} |\frac{\partial u}{\partial \nu}|^2.
\end{aligned}$$

Using Wirtinger inequality and setting $\lambda = 2r$ we deduce that

$$\begin{aligned}
\int_{I_j} |u - m_j| \left| \frac{\partial u}{\partial r} \right| d\sigma &\leq \frac{4r^2}{2\lambda} \int_{I_j} |u_\tau|^2 + \frac{\lambda}{2} \int_{I_j} |\frac{\partial u}{\partial \nu}|^2 \\
&\leq r \int_{I_j} |u_\tau|^2 + r \int_{I_j} |\frac{\partial u}{\partial \nu}|^2 \\
&= r \int_{I_j} |\nabla u|^2.
\end{aligned}$$

Finally summing over j, we get (27) and the monotonicity is proved.

The last conclusion of the proposition then follows from the case of equality in the above inequalities. □

Remark 8. *We have used the basic inequality $\mathcal{H}^1(I_i) \leq 2\pi r$ in our monotonicity proof. But if we assume $\mathcal{H}^1(I_i) \leq r\pi(1+\delta)$ then we obtain a much better estimate, namely, that*

$$r \mapsto \frac{E(r)}{r^{1+\alpha}}$$

is non decreasing, with $\alpha = \frac{1-\delta}{1+\delta} > 0$. This will be one of our crucial ingredient in getting C^1 estimates.

3.2.3 An extension tool

Our second tool at our disposal to prove the regularity of the singular set will be the following extension estimate.

Lemma 8. *Let $w \in W^{1,2}(\partial B(0,1))$ and let $h \in W^{1,2}(B(0,1))$ be its harmonic extension (i.e. the unique harmonic function in the ball that concides with w on the circle). Then*

$$\int_{B(0,1)} |\nabla h|^2 dx \leq \int_{\partial B(0,1)} \left| \frac{\partial w}{\partial \tau} \right|^2 d\mathcal{H}^1.$$

Proof. Let expend w in Fourier series. We get

$$w = \sum_{n\in\mathbb{N}} a_n \cos(n\theta) + b_n \sin(n\theta).$$

Remembering the Laplacian operator in polar coordinates,

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2},$$

we easily check that the following function

$$\sum_{n\in\mathbb{N}} r^n a_n \cos(n\theta) + r^n b_n \sin(n\theta)$$

is harmonic and coincides with w on the circle. By uniqueness, it must be equal to h. Now we compute

$$\int_{B(0,1)} |\nabla h|^2 dx = \pi \sum_{n\in\mathbb{N}} n(a_n^2 + b_n^2)$$

and

$$\int_{\partial B(0,1)} |\frac{\partial w}{\partial \tau}|^2 = \pi \sum_{n\in\mathbb{N}} n^2(a_n^2 + b_n^2),$$

from which we deduce the lemma. □

The above lemma is a convenient tool to create some competitors.

Lemma 9. *Let $I \subset \partial B_r$ be an arc of circle such that*

$$r(\pi - 10^{-3}) \leq \mathcal{H}^1(I) \leq (\pi + 10^{-3})r$$

and let $w \in H^1(I)$. Then there exists some $v \in H^1(B_r)$ such that $v = g$ on I and

$$\int_{B_r} |\nabla v|^2 dx \leq \frac{C}{r} \int_I |\frac{\partial w}{\partial \tau}|^2 d\mathcal{H}^1. \tag{31}$$

Proof. We first consider a bi-Lipschitz mapping from I to a half circle S, the Lipschitz constant being uniform for all I satisfying (31). Then we extend it symmetrically to the entire circle and we apply the Lemma before on the harmonic extension. □

3.2.4 The C^1 regularity proof

We now prove some C^1 regularity for the minimizers of Mumford-Shah.

Notation:

$$\omega_u(x,r) := \frac{1}{r} \int_{B(x,r)\setminus K} |\nabla u|^2 \, dx.$$

We will sometimes simply write $\omega(x,r)$.

With all the preliminary tools at hand we are now ready to prove the following statement.

Theorem 3.2. *Let (u, K) be a minimizer of* (25)*. Then if $x \in K$ and $r > 0$ are such that $B(x,r) \subset \Omega$ and*

$$\beta_K(x,r) + \omega_u(x,r) \leq 10^{-5},$$

then $K \cap B(x, r/10)$ is a $C^{1,\alpha}$ regular curve.

Ideas of proof. We follow the same approach as for the C^1 regularity of almost minimizers. We start with a ball where $\beta_K(x,r) + \omega_u(x,r) \leq 10^{-5}$ and construct a first competitor with "walls" to get a density estimate. Before that, and because of the Dirichlet term, we first select a good radius $s \in (r/2, r)$ such that

$$\int_{\partial B_s \setminus K} |\nabla u|^2 d\mathcal{H}^1 \leq \frac{2}{r} \int_{B_r} |\nabla u|^2 dx,$$

which is always possible. Then we replace K by the competitor L made of a diameter of $B(x,s)$, union two vertical walls, as we did for almost minimizers (proof of Proposition 4). Then we use the extension Lemma 9 to define a suitable competitor v for the function u, well defined in $B_s \setminus L$ and equal to u outside B_s. This provides the estimate

$$\mathcal{H}^1(K \cap B_s) \leq 2s + Cr(\beta(x,r) + \omega(x,r)) \leq 2s(1 + 10^{-4}).$$

This is enough to find some radius $s' \in (r/4, r/2)$ which intersects the singular set K by exactly 2 opposite points $\{z, z'\}$. We then can proceed with a similar competitor, without the walls, which provides now the better estimate

$$\mathcal{H}^1(K \cap B_{s'}) \leq |z - z'| + Cr(\omega(x,r))$$

from which, by the fundamental height estimate (Lemma 5) easily yields

$$\beta(x, s') \leq C\omega(x,r).$$

We then use the monotonicity formula which provides a decay of $\omega(x,r)$ by means of r^α to deduce some $C^{1,\alpha}$ estimate on K. $\square$

4 Lecture Day ♯4: Blow up limits and global minimizers

Before talking about blow-up limits of Mumford-Shah minimizers, we first look at blow-up limits for an almost minimal set in 2D.

4.1 Blow-up limits of planar 1D-almost minimal sets

4.1.1 Monotonicity of density

We want to prove a monotonicity on the density ratio $r \mapsto \frac{\mathcal{H}^1(K\cap B(x_0,r))}{r}$, when K is an almost minimizer. For that we need to estimate the derivative of the function

$$\ell(r) := \mathcal{H}^1(K \cap B(x_0, r)).$$

The function ℓ is nondecreasing, thus in particular it is a BV function. Moreover, it is absolutely continuous. Let try to give some bounds on the derivative. Firstly from Eilenberg's inequality

$$\int_0^r \sharp\{\partial B(x_0,t) \cap E\}dt \leq \mathcal{H}^1(E \cap B(x_0, r))$$

we obtain a first bound

$$\sharp\{\partial B(x_0,t) \cap K\} \leq \ell'(r)$$

for almost every r. Let us be more precise. From the coarea applied with the function $\varphi : x \mapsto |x - x_0|$ and the rectifiable set $E \cap B(x,r)$ we get the equality

$$\int_{E\cap B(x_0,r)} f(x)c(x)d\mathcal{H}^1(x) = \int_{\mathbb{R}} \left(\int_{\varphi^{-1}(t)\cap E\cap B(x_0,r)} f(x)d\mathcal{H}^0 \right) dt,$$

where $c(x)$ is the coarea factor, which is equal here to

$$c(x) = \left| \frac{x - x_0}{|x - x_0|} \cdot \tau(x) \right|,$$

where $\tau(x)$ is a unit tangent vector to K at point x, that exists $\mathcal{H}^1$ a.e. In other words, $c(x)$ is the cosine of the angle between the tangent vector at x and the normal direction $x - x_0$. Denoting this angle by $\theta(x)$ we define have proved the inequality

$$\int_{E\cap B(x_0,r_2)\setminus B(x_0,r_1)} |\cos(\theta(x))|d\mathcal{H}^1(x) = \int_{r_1}^{r_2} \sharp\{\partial B(x_0,t) \cap E\}dt.$$

Now let ν be the measure

$$\nu := \cos(x)\mathcal{H}^1|_E$$

and let $f(r) := \nu(E \cap B(x_0, r))$. From the above we find that

$$\sharp\{\partial B(x_0,t) \cap K\} \leq f'(r) \leq \ell'(r). \tag{32}$$

We can now prove the following monotonicity result.

Proposition 9. *Assume that K is an almost minimal set in $\Omega \subset \mathbb{R}^2$ with gauge function $h : \mathbb{R}^+ \to \mathbb{R}^+$. Then for every $x_0 \in K$ and $r \in (0, \min(diam(K), dist(x_0, \partial\Omega))$ we have that*

$$d : r \mapsto \frac{\mathcal{H}^1(K \cap B(x_0, r))}{r} + \int_0^r \frac{h(t)}{t} dt$$

is non decreasing. Moreover, if $h = 0$ and $d(r)$ is constant then K must be a cone.

Proof. We know that $\sharp\{\partial B(x_0, t) \cap K\} < +\infty$ for $\mathcal{H}^1$-a.e. r. We then pick such a radius r and compare K in $B(x_0, r)$ with the competitor made of the union of segments joining x_0 to each point of K on the circle, namely,

$$K' := (K \setminus B(x_0, r)) \bigcup_{x \in \partial B(x_0, r)} [x_0, x].$$

Comparing K with K' yields

$$\mathcal{H}^1(K \cap B(x_0, r)) \leq \mathcal{H}^1(K' \cap B(x_0, r)) + rh(r) \leq r\sharp\{\partial B(x_0, t) \cap K\} + rh(r).$$

Using now (32) it comes

$$\ell(r) \leq r\sharp\{\partial B(x_0, r) \cap K\} + rh(r) \leq rf'(r) + rh(r) \leq r\ell'(r) + rh(r), \tag{33}$$

from which we get the monotonicity result. Now assuming $h = 0$ and $\frac{\ell(r)}{r}$ is contant, we deduce that the above inequalities in (33) must be all equalities (with h=0). In particular $\cos(\theta(x)) = 1$ for all x which shows that K must be a cone and which finishes the proof of the proposition. □

4.1.2 Classification of blow-ups for 1D-minimal sets

We are now ready to classify the possible blow-up limits of a planar 1D-almost minimal set. First we observe that due to the monotonicity behavior of density we know that the following limit exists for all $x_0 \in K$:

$$\lim_{r \to 0} \frac{\mathcal{H}^1(K \cap B(x_0, r))}{r} = \ell_0.$$

Now let $r_n \to 0$ and let

$$K_n := \frac{1}{r_n}(K - x_0).$$

The set K_n is almost minimizing in $\frac{1}{r_n}(\Omega - x_0) \to \mathbb{R}^2$, with associated gauge function $h(r_n \cdot)$ (which converges to zero). It can be proved that K_n converges (in some local Hausdorff sense) to some set K_0 in $\mathbb{R}^2$ which satisfies the following facts:

- K_0 is an almost minimal set in $\mathbb{R}^2$ (i.e. with $h = 0$)
- $r \mapsto \frac{\mathcal{H}^1(K_0 \cap B(0,r))}{r}$ is constant, equal to ℓ_0
- K_0 is a cone, because the proof of monotonicity says that $d' = 0$ thus $\frac{1}{c(x)} = 1$ for all x.

- we deduce that K_0 is entirely caracterized by its intersection with $\partial B(0,1)$, and must be, either a line, or 3 half-lines meeting with 120 degree. Indeed, the intersection cannot have only 2 points unless they are exactly on a diameter (otherwise we can easily construct a better competitor), and it cannot have more than 3 points otherwise there always exist at least two making an angle smaller than 120 degree which means that a better competitor is possible.

Remark 9. *For higher dimensions, the classification of blow-up limits is known in dimension 3, due to a result by Jean Taylor [Tay76]. Some minimal cones of dimension 2 in $\mathbb{R}^4$ was also recently found by Xiangyu Liang in [Xia15, Xia14, Xia13].*

4.2 Blow-up limits of Mumford-Shah minimizers in dimension 2

For Mumford-Shah minimizers we can also proceed to a blow-up process, by defining

$$u_r := \frac{1}{\sqrt{r}} u(rx + x_0) \qquad K_r := \frac{1}{r}(K - x_0),$$

and then let $r \to 0$. Bonnet proved that the couple (u_r, K_r) converges, in some sense, to some couple (u, K) satisfying a certain minimization property (sort of local Mumford-Shah minimizer in the whole plane). He called this a "global minimizer" and was able to completely characterize them under a connected assumption.

Theorem 4.1 (Classification of connected global minimizers in $\mathbb{R}^2$)**.** *Let (u, K) be a global minimizer in $\mathbb{R}^2$ such that K is connected. Then it belongs to the following list.*

1. *$K = \emptyset$ and u is constant.*
2. *(Line) K is a line and u is constant on each side.*
3. *(Propeller) K is the union of three half-lines meeting at their tip by angles of 120 degree.*
4. *(Cracktip) Up to translation, rotation, or additional constant, K is a half line and u is equal to the cracktip function defined in* (**??**)*.*

Elements of proof. Let (u, K) be a global minimizer, and assume that K is connected. The key ingredient is the monotonicity formula. It says that the quantity $\varphi(r) = E(r)/r$ is nondecreasing (Proposition 8). Thus, let $\varphi(+\infty)$ and $\varphi(0)$ be the respective limits for r going to 0 and $+\infty$. First we notice that $\varphi(+\infty)$ is finite, due to the estimate $\int_{B_r} |\nabla u|^2 dx \leq 2\pi r$ valid for any global minimizer. Next, we use the blow-up and blow-in procedure, to obtain at the limit two new pairs (u_0, K_0) and (u_∞, K_∞) which are again global minimizers with connected singular sets, for which their respective quantity $\varphi(r)$ is constantly equal to $\varphi(\infty)$ for the blow-in and $\varphi(0)$ for the blow-up. By the last conclusion of the monotonicity result (Proposition 8), it follows that $\varphi(\infty)$ and $\varphi(0)$ can be only equal to 0 or 1. Indeed, in the case when it is not 0, using the last conclusion of Proposition 8 we deduce that u is of the form $C(r) + \alpha(r)\sin(\frac{\theta - \theta(r)}{2})$ for some $C(r)$, $\alpha(r)$ and $\theta(r)$. But since u is harmonic it follows that $\theta(r) = \theta_0$, $\alpha(r) = \alpha\sqrt{r}$ and $C(r) = C$. Finally, the constant α must be equal to $\sqrt{2/\pi}$ due to the fact that we have a Mumford-Shah minimizer, which implies some variational equalities (see [Dav05, Page 406]), leading to $\varphi(r) = 1$.

Now we analyse two cases. The first case is when $\varphi(\infty)$ is equal to 0, then returning to the original minimizer u, the monotonicity says that $\varphi(r) = 0$ all the time, thus $\nabla u = 0$ and K locally minimizes the length $\mathcal{H}^1$ among topological competitors. This implies that K must be one of the sets described in 1. 2. and 3. of the statement.

The most delicate case is when $\varphi(\infty) = 1$. First notice that this value does not change by changing the origin (i.e. the point at which balls are centered in the computation of φ). Then, by a non trivial contradiction argument, it is possible to find in K a point at witch $\varphi(0) = 1$. Looking now at this point we obtain that $\varphi(r)$ is constant, equal to 1, and the same argument as the one used just before says that u is of the type described in 4. □

Remark 10. *One way to prove the Mumford-Shah conjecture would be to improve the result of Theorem 4.1, by proving the same statement as Theorem 4.1 without assuming connectedness of K. Some work in this direction has been done in [DL02].*

Let us mention that the "finite number of curves" in the statement of Bonnet's result, follows from the classification of blow-up limits, as explained below.

Elements of proof for Theorem 3.1. Let us describe the ingredients to deduce Theorem 3.1 from the classification of blow-up limits. One first thing is to prove the finite number of pieces. For this purpose Bonnet first proves that, even if the blow-up limit at one point may not be unique in general, it is always of same type, which allows him to classify points with respect to the type of the blow-ups (regular point, cracktip, or triple point). Then he is able to prove that there is a finite number of triple points. Indeed, assuming that a sequence of triple points T_n accumulates onto some point P, he gets a contradiction by considering, basically, the blow up limit in $B(T_n, 2|T_n - T_{n+1}|)$ which would converge to some fancy global minimizer with two triple points. Indeed, it is not difficult to see that a blow-up limit with "moving" center will also converge to one of the short list of global minimizer. Now if a sequence of triple points accumulates, one can construct a certain blow-up (with "moving center", centered at the sequence of triple points) whose density at the limit does not correspond to any of the list of Theorem 4.1, because it would have a density of two triple points (in reality the blow-up sequence $B(T_n, 2|T_n - T_{n+1}|)$ may not really work because one should also take into consideration the speed of convergence to the respective propeller at T_n and T_{n+1}, but the idea is roughly the same). This implies the finite number of endpoints as well.

Then the C^1-regularity relies on the fact that, if x_0 is a point at which the blow-up limits are a lines, then there exits r_0 such that K is almost flat in $B(x_0, r)$ for all $0 < r < r_0$ and intersects $\partial B(x_0, r)$ on both sides for many $r > 0$. In this situation the monotonicity Lemma can be straighten with a greater power, which implies that $r \mapsto \int_{B(x_0,r)} |\nabla u|^2 dx$ behaves like $Cr^{1+\alpha}$ (because the arc of circles are smaller, which lead to a better Poincaré-Wirtinger constant). This, morally says that K is an almost-minimizing set for $\mathcal{H}^1$, with excess of minimality controlled by $Cr^{1+\alpha}$. It is then classical to get $C^{1,\alpha/2}$ estimates from this fact. □

4.3 Blow-up limits of Mumford-Shah minimizers in dimension 3

We finish this lecture with some results in dimension 3. Actually, the classification of blow-up limits for a Mumford-Shah minimizer in dimension 3 is still an open problem. However, some partial facts are known. We refer to [Lem09] for more details.

For instance, the following nice question of Guy David is still open: let (u, K) be a global Mumford-Shah minimizer in $\mathbb{R}^3$ and assume that K is contained in a plane. Is it true that K must be the whole plane or a half-plane ? This question was partially solved in [Lem14], showing that if K is contained in a half-plane then it must be the whole half-plane.

A last, let us mention the work in [AH18] where a nice question about the global minimizer "cracktip$\times\mathbb{R}$" in $\mathbb{R}^3$ is studied.

5 Lecture day ♯5: Regularity for connected minimizers of the Griffith functional

In this section I presented the proof of the main result contained in [JFFA19] and I refer directly to the paper for more details.

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